Random Matrix Theory Assignment 2

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Due date: 26th of May 2020

Q1. Involutions and algebra... lots and lots of algebra

1.1) Standard properties of involutions

We first note that any projective involution $I : \mathrm{Hom}_{\mathbb{R}}(\mathcal{H}) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{H})$, regardless of classification, has ring homomorphism properties. Hence using similar arguments to elementary algebra calculations we have for any projective involution I that

$$\mathrm{id}_{\mathcal{H}} = I(\mathrm{id}_{\mathcal{H}})^{-1} I(\mathrm{id}_{\mathcal{H}}) = I(\mathrm{id}_{\mathcal{H}})^{-1} I(\mathrm{id}_{\mathcal{H}}\mathrm{id}_{\mathcal{H}}) = I(\mathrm{id}_{\mathcal{H}})^{-1} I(\mathrm{id}_{\mathcal{H}}) I(\mathrm{id}_{\mathcal{H}}) = I(\mathrm{id}_{\mathcal{H}})\,. \quad (1.1.1)$$

Equivalently, from Proposition 3.5, we have that $I(X) = UB(X)U^{-1}$ for some $U \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$ (dependent on the classification of the projective involution, see below table) and an operator B that satisfies $B(\mathrm{id}_{\mathcal{H}}) = \mathrm{id}_{\mathcal{H}}$, i.e conjugation, transposition, etc. and so

$$I(\mathrm{id}_{\mathcal{H}}) = UB(\mathrm{id}_{\mathcal{H}})U^{-1} = UU^{-1} = \mathrm{id}_{\mathcal{H}}\,. \quad (1.1.2)$$

Using this, we also see that for an automorphic projective involution I and invertible $X \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{H})$,

$$\mathrm{id}_{\mathcal{H}} = I(\mathrm{id}_{\mathcal{H}}) = I(XX^{-1}) = I(X)I(X^{-1}) \quad \text{so} \quad I(X^{-1}) = I(X)^{-1}\,. \quad (1.1.3)$$

It is clearly trivial that this is the case for an anti-automorphic projective involution too, hence the property holds for any projective involution.

1.2) Similarity of involutions of same kind

We first make good on the claim preceding (1.1.2) where we can translate Proposition 3.5 into the following table for $I(X) = UB(X)U^{-1}$ with $X \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{H})$:

Classification	U	B	Restrictions
Automorphic, linear	$W^{-1}\gamma_5 W$	id(X)=X	$W \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$ and $\gamma_5 = \mathrm{diag}(\mathbb{1}_{n/2}, -\mathbb{1}_{n/2})$
Automorphic, anti-linear ($\mathbb{R}$)	e^{-iW}	$K(X) = X^*$	$W \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{H})$
Automorphic, anti-linear ($\mathbb{H}$)	$e^{-iW^*/2}\hat{\tau}_2 e^{-iW/2}$	$K(X) = X^*$	$W \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{H})$
Anti-automorphic, linear ($\mathbb{R}$)	WW^T	$T(X) = X^T$	$W \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$
Anti-automorphic, linear ($\mathbb{H}$)	$W\hat{\tau}_2 W^T$	$T(X) = X^T$	$W \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$
Anti-automorphic, anti-linear	$W^\dagger\gamma_5\lvert\Lambda\rvert W$	$A(X) = X^\dagger$	$W \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$ and γ_5 as above

In all cases W is a bijection.

Now suppose I_1 and I_2 are projective involutions of the same kind. Then

$$I_1(X) = U_1 B(X) U_1^{-1} \quad \text{and} \quad I_2(X) = U_2 B(X) U_2^{-1}\,. \tag{1.2.1}$$

Rearranging for $B(X)$ we see that

$$U_1^{-1} I_1(X) U_1 = U_2^{-1} I_2(X) U_2\,, \quad \text{so} \quad I_1(X) = U_1 U_2^{-1} I_2(X) U_2 U_1^{-1}\,,$$

and so clearly in defining $U = U_1 U_2^{-1} \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$ (with $U^{-1} = U_2 U_1^{-1}$) we see that $I_1(X) = U I_2(X) U^{-1}$, meaning any projective involution is unique up to similarity.

To show the next property, we first analyse the simplest case - an automorphic linear projective involution. Let $U = U_1 U_2^{-1} = W_1^{-1}\gamma_5 W_1 W_2^{-1}\gamma_5 W_2$. Then we have (where $B(X) = \mathrm{id}(X)$):

$$\begin{aligned} I_1(U) &= (W_1^{-1}\gamma_5 W_1)(W_1^{-1}\gamma_5 W_1 W_2^{-1}\gamma_5 W_2)(W_1^{-1}\gamma_5 W_1) \\ &= (W_2^{-1}\gamma_5 W_2)(W_1^{-1}\gamma_5 W_1) = U^{-1}\,. \end{aligned} \tag{1.2.2}$$

Of course, we also then have $I_2(U) = U I_1(U) U^{-1} = U^{-1}$. We see here that the property of our similarity matrices that gives us this desired effect is that U_1 is self-inverse. Indeed, for the two other automorphisms, we note that $U_1^{-1} = K(U) = U^*$. Hence we can perform an identical calculation, for example the (harder) quarternion anti-linear automorphism:

$$\begin{aligned} I_1(U) &= e^{-iW_1^*/2}\hat{\tau}_2 e^{-iW_1/2}(e^{-iW_1^*/2}\hat{\tau}_2 e^{-iW_1/2} e^{iW_2/2}\hat{\tau}_2 e^{iW_2^*/2})^* e^{iW_1/2}\hat{\tau}_2 e^{iW_1^*/2} \\ &= (e^{-iW_1^*/2}\hat{\tau}_2 e^{-iW_1/2})(e^{iW_1/2}\hat{\tau}_2 e^{iW_1^*/2})(e^{-iW_2^*/2}\hat{\tau}_2 e^{-iW_2/2} e^{iW_1/2}\hat{\tau}_2 e^{iW_1^*/2}) \\ &= e^{-iW_2^*/2}\hat{\tau}_2 e^{-iW_2/2} e^{iW_1/2}\hat{\tau}_2 e^{iW_1^*/2} = U_2 U_1^{-1} = U^{-1}\,. \end{aligned} \tag{1.2.3}$$

We can then perform a similar calculation on an anti-automorphism, say linear over the reals for simplicity. Then

$$\begin{aligned} I_1(U) = I_1(W_1 W_1^T (W_2^T)^{-1} W_2^{-1}) &= W_1 W_1^T (W_1 W_1^T (W_2^T)^{-1} W_2^{-1})^T (W_1^T)^{-1} W_1^{-1} \\ &= W_1 W_1^T (W_2^T)^{-1} W_2^{-1} W_1 W_1^T (W_1^T)^{-1} W_1^{-1} \\ &= U_1 U_2^{-1} = U\,. \end{aligned}$$

Of course we could sit here longer and calculate all of them, but it is worth noting that Schur's Lemma is what unifies all of this. I couldn't quite nut out the details to prove it in this more elegant way, but I think it has something to do with noticing that $\mathrm{id} = r : \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$ can be used since $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H})$ is an irreducible representation of itself. With all of this info in mind, this is enough to show the desired outputs in the question.

Suppose we have $U^2 = z\,\mathrm{id}_{\mathcal{H}}$ for $z \in \mathbb{R}_{\neq 0}$. Then we have

$$(I_1 \circ I_2)(X) = U I_2^2(X) U^{-1} = U X U^{-1}\,, \tag{1.2.4}$$

$$\text{so} \quad (I_1 \circ I_2)^2(X) = (I_1 \circ I_2)(U X U^{-1}) = U^2 X U^{-2} = (z\,\mathrm{id}_{\mathcal{H}})\; X\; \left(\frac{1}{z}\,\mathrm{id}_{\mathcal{H}}\right) = X\,. \tag{1.2.5}$$

Hence, under these assumptions, $I_1 \circ I_2$ is an involution. This is equivalent to I_1 and I_2 commuting since if this is true then

$$\begin{aligned} (I_1 \circ I_2)(X) = U X U^{-1} = U^{-1} X U = (I_2 \circ I_1)(X)\,, \\ \text{so} \quad U^2 X U^{-2} = X\,, \quad \text{so} \quad U^2 = z\,\mathrm{id}_{\mathcal{H}} \text{ is a possibility}\,. \end{aligned} \tag{1.2.6}$$

1.3) QCD Dirac operator D

Now consider the two dimensional QCD Dirac operator D corresponding to the gauge group SU(2) in its adjoint representation. This operator satisfies the symmetries

$$D = -D^\dagger = -D^T = -\hat{\tau}_3 D \hat{\tau}_3 \tag{1.3.1}$$

meaning it is anti-Hermitian, anti-symmetric and chiral. With the second equality, we can see that this gives us $D^\dagger = D^T$, so $D^* = D$ meaning D is a real matrix - this is its reality condition.

When putting D on a two-dimensional lattice, there are two possible additional symmetris of D obtained via anti-commutation relations:

a) either we have a Hermitian orthogonal matrix Γ with $\Gamma^2 = \mathbb{1}_{2N}$, $\mathrm{tr}\Gamma = 0$ and $[\hat{\tau}_3, \Gamma]_+ = [D, \Gamma]_+ = 0$,

b) or we have two Hermitian orthogonal matrices Γ_1, Γ_2 with $\Gamma_j^2 = \mathbb{1}_{2N}$, $\mathrm{tr}\Gamma_j = 0$, $\mathrm{tr}\hat{\tau}_3\Gamma_1\Gamma_2 = 0$ and $[\hat{\tau}_3, \Gamma_j]_+ = [D, \Gamma_j]_+ = [\Gamma_1, \Gamma_2]_+ = 0$ (where $j = 1, 2$).

Our aim here is to bring D into its smallest diagonal block structure to analyse to which symmetric space it belongs.

We first investigate Γ with the properties of a). Since D is a real matrix and Γ is Hermitian, we use the anti-commutation of these two to see how to diagonalise Γ. We have

$$[D, \Gamma]_+ = 0\,, \quad \text{so } -D = \Gamma D \Gamma\,, \quad \text{so } -D^\dagger = \Gamma^\dagger D^\dagger \Gamma^\dagger\,, \quad \text{and } -D^T = \Gamma^T D^T \Gamma^T\,. \tag{1.3.2}$$

Using our reality condition from (1.3.1), we then see that this implies

$$\Gamma^\dagger D \Gamma^\dagger = \Gamma^T D \Gamma^T\,, \tag{1.3.3}$$

and so in comparing terms we see $\Gamma^\dagger = \Gamma^T$ and so Γ must also be a real matrix. Hence since Γ is a real Hermitian matrix, thus a real symmetric matrix, we can diagonalise Γ by an orthogonal $U \in \mathrm{O}(2N)$. Then, since $\Gamma^2 = \mathbb{1}_{2N}$ is an involution, it must have eigenvalues of ± 1 and because $\mathrm{tr}\,\Gamma = 0$ the multiplicity of these eigenvalues must be the same. Hence we choose $\Gamma = U\hat{\tau}_3 U^T$. Then, since $[\hat{\tau}_3, \Gamma]_+ = 0$, this tells us that Γ must have an off-diagonal structure (this is due to the Pauli matrix anti-commutation relations - see (1.3.12) for an example of this calculation). We can exploit this off-diagonal structure, and the Hermiticity and orthogonality of Γ to write

$$\Gamma = \begin{pmatrix} 0 & V \\ V^T & 0 \end{pmatrix} \qquad \text{where } V \in \mathrm{O}(N)\,. \tag{1.3.4}$$

This then allows us to write a new orthogonal matrix $U' = U\mathrm{diag}(V^T, \mathbb{1}_N) \in \mathrm{O}(2N)$. We quickly verify that this is indeed orthogonal since

$$\det(U') = \det(U)\det(\mathrm{diag}(V^T, \mathbb{1}_N)) = 1\,, \quad \text{and also} \tag{1.3.5}$$

$$U'U'^T = U\mathrm{diag}(V^T, \mathbb{1}_N)\mathrm{diag}(V, \mathbb{1}_N)U^T = U\mathrm{diag}(V^T V, \mathbb{1}_N)U^T = UU^T = \mathbb{1}_{2N}\,, \tag{1.3.6}$$

so U' is well defined.

From this we can then write this off-diagonal structure more explicitly as $U'^T \Gamma U' = \hat{\tau}_1$.

We can do a similar procedure for the properties of b). All of the same derivations as above hold which leaves us with $\Gamma_1 = U\hat{\tau}_3 U^T$ and $U'^T\Gamma_1 U' = \hat{\tau}_1$. Using the fact that $[\Gamma_1, \Gamma_2]_+ = 0$, we can rewrite this as $[\hat{\tau}_1, U'^T\Gamma_2 U']_+ = 0$, meaning that $U'^T\Gamma_2 U' = \tau_2 \otimes \tilde{V}$ for $\tilde{V} \in \mathrm{O}(N)$. Using the fact that $\Gamma^2 = \mathbb{1}$ we can then calculate:

$$\begin{aligned}
(U'^T\Gamma_2 U')(U'^T\Gamma_2 U') &= (\tau_2 \otimes \tilde{V})(\tau_2 \otimes \tilde{V})\,, \\
\text{so} \quad U'^T\Gamma_2^2 U' &= \tau_2^2 \otimes \tilde{V}^2\,, \\
\text{so} \quad \mathrm{diag}(\mathbb{1}_N, \mathbb{1}_N) &= \mathbb{1}_2 \otimes \tilde{V}^2\,, \\
\text{so} \quad \tilde{V}^2 &= \mathbb{1}_N\,.
\end{aligned} \tag{1.3.7}$$

Hence we see that $\tilde{V}$ is self inverse. Further to this, we can also calculate

$$\begin{aligned}
\mathrm{tr}\hat{\tau}_3\Gamma_1\Gamma_2 &= \mathrm{tr}\,\hat{\tau}_3(U'\hat{\tau}_1U'^T)(U'(\tau_2 \otimes \tilde{V})U'^T) \\
&= \mathrm{tr}\,U'^T(\tau_3 \otimes \mathbb{1}_N)U'(\tau_1 \otimes \mathbb{1}_N)(\tau_2 \otimes \tilde{V}) \\
&= \mathrm{tr}\,(\tau_3 \otimes \mathbb{1}_N)(-i\tau_3 \otimes \tilde{V}) \\
&= -i\,\mathrm{tr}\,(\tau_3^2 \otimes \tilde{V}) \\
&= -i\,\mathrm{tr}\mathbb{1}_2\,\mathrm{tr}\tilde{V} = -2i\,\mathrm{tr}\tilde{V} = 0
\end{aligned} \tag{1.3.8}$$

where in the 3rd line we used the fact that since $(\tau_3 \otimes \mathbb{1}_N) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, we can write $U'^T(\tau_3 \otimes \mathbb{1}_N)U' = (\tau_3 \otimes \mathbb{1}_N)$ which can be easily seen from writing out the matrix multiplication and using the fact that the columns of U' satisfy $u_i u_j = \delta_{i,j}$ since they are orthonormal columns. Therefore, we have determined that we can diagonalise $\tilde{V} = \hat{V}\gamma_5\hat{V}^T$ for $\hat{V} \in O(2N)$ where $\gamma_5 = \mathrm{diag}(\mathbb{1}_N, -\mathbb{1}_N)$. We can now define the orthogonal matrix $\hat{U} = U'(\mathbb{1}_2 \otimes \hat{V})$, which can be verified is orthogonal with a similar calculation to part a) which gives us

$$\hat{U}^T\Gamma_1\hat{U} = \hat{\tau}_1 \quad \text{and} \quad \hat{U}^T\Gamma_2\hat{U} = i\tau_2 \otimes \gamma_5\,. \tag{1.3.9}$$

The factor of i arises in order to keep everything real since $\tau_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$.

Returning to the simpler part a), we can exploit the relation $[D, \Gamma]_+ = 0$ to calculate a relation on UDU^T as follows:

$$\begin{aligned}
UDU^T = U(-\Gamma D\Gamma)U^T = -(U\Gamma U^T)(UDU^T)(U\Gamma U^T) &= -\hat{\tau}_1 UDU^T D\hat{\tau}_1\,, \\
\text{so} \quad [\hat{\tau}_1, UDU^T]_+ &= 0\,.
\end{aligned} \tag{1.3.10}$$

We also know that $[\hat{\tau}_3, D]_+ = 0 = [U\hat{\tau}_3U^T, UDU^T]_+ = [\hat{\tau}_3, UDU^T]$ (using the same derivation as above). We start by analysing the initial block form

$$UDU^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{where } A, B, C, D \in \mathbb{R}^{n \times n}\,. \tag{1.3.11}$$

Then, because of the $\hat{\tau}_3$ anti-commutativity, we have

$$\begin{aligned}
(\tau_3 \otimes \mathbb{1}_N)\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}(\tau_3 \otimes \mathbb{1}_N) = 0 \\
\text{so} \quad \begin{pmatrix} 2A & 0 \\ 0 & -2D \end{pmatrix} = 0\,, \quad \text{so} \quad A = D = 0\,.
\end{aligned} \tag{1.3.12}$$

Then $\hat{\tau}_1$ anti-commutativity gives us

$$\begin{aligned}
(\tau_1 \otimes \mathbb{1}_N) \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} (\tau_1 \otimes \mathbb{1}_N) &= 0 \\
\text{so} \quad \begin{pmatrix} B+C & 0 \\ 0 & B+C \end{pmatrix} &= 0\,, \quad \text{so} \quad C = -B\,.
\end{aligned} \tag{1.3.13}$$

Because $D = -D^T$, this also implies that $(UDU^T)^T = -UDU^T$, hence we must have $B = B^T$. Putting all of this together, we arrive at:

$$UDU^T = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} = i\tau_2 \otimes B\,, \quad \text{where } B \in \mathrm{Sym}_{\mathbb{R}}(n)\,. \tag{1.3.14}$$

In part b), Γ_1 gives us the same form but the addition of Γ_2 symmetry will put further conditions on the block structure of B. We have the additional symmetry of $\hat{U}^T\Gamma_2\hat{U} = i\tau_2 \otimes \gamma_5$, so calculating the same way as in (1.3.10) this gives $[i\tau_2 \otimes \gamma_5, \hat{U}^T\Gamma_2\hat{U}]_+ = 0$. We can write $\hat{U}^T\Gamma_2\hat{U} = i\tau_2 \otimes \hat{B}$ and then calculate

$$\begin{aligned}
(i\tau_2 \otimes \gamma_5)\left(i\tau_2 \otimes \hat{B}\right) + \left(i\tau_2 \otimes \hat{B}\right)(i\tau_2 \otimes \gamma_5) &= 0 \\
\text{so} \quad \mathbb{1}_2 \otimes \left(\gamma_5\hat{B} + \hat{B}\gamma_5\right) &= 0\,,
\end{aligned} \tag{1.3.15}$$

which tells us that $\hat{B}$ must be chiral (since $[\gamma_5, \hat{B}]_+$=0)! Therefore, for our conditions in part b), we arrive at the final form

$$\hat{U}D\hat{U}^T = i\tau_2 \otimes \begin{pmatrix} 0 & W \\ -W^T & 0 \end{pmatrix}\,, \qquad \text{where } W \in \mathbb{R}^{n/2\times n/2}\,. \tag{1.3.16}$$

Therefore, in both cases we see that D is a representation of the class of real anti-symmetric chiral matrices. It is possible that there is an extra word to describe the added layer of chirality in the second case but I was not able to find this word. Nevertheless the first statement remains true and so we have our matrix symmetries. $\square$

Q2. Lebesgue measures on matrix spaces

2.1) Vandermonde determinant

We will first show that the Vandermonde determinant,

$$\Delta_N(\Lambda) = \prod_{1\le l<m\le N} (\lambda_m - \lambda_l) = \det[\lambda_a^{b-1}]_{a,b=1,\dots,N}\,, \tag{2.1.1}$$

$$\text{where}\quad \det[\lambda_a^{b-1}]_{a,b=1,\dots,N} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{N-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \dots & \lambda_N^{N-1} \end{pmatrix}, \tag{2.1.2}$$

is indeed a determinant. We will do this via a complete induction, where $S(N)$ is the statement in (2.1.1).

We first check the base case, where we note that $\Delta_N(\Lambda)$ is not well defined for $N = 1$ so we check $S(2)$:

$$S(2) : \Delta_2(\Lambda) = \prod_{1\le l<m\le 2} (\lambda_m - \lambda_l) = \lambda_2 - \lambda_1 = \det\begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} = \det[\lambda_a^{b-1}]_{a,b=1,2}\,, \tag{2.1.3}$$

so the base case holds as required. We then form the inductive hypothesis: assume $S(1), S(2), \dots, S(N-1)$ holds for some $k \in \mathbb{N}_{>2}$. We now want to show that $S(N)$ holds. We can subtract the Nth row from all other $(N-1)$ rows (hence preserving the determinant) to get

$$\det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{N-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N-1} & \lambda_{N-1}^2 & \dots & \lambda_{N-1}^{N-1} \\ 1 & \lambda_N & \lambda_N^2 & \dots & \lambda_N^{N-1} \end{pmatrix} = \det\begin{pmatrix} 0 & \lambda_1 - \lambda_N & \lambda_1^2 - \lambda_N^2 & \dots & \lambda_1^{N-1} - \lambda_N^{N-1} \\ 0 & \lambda_2 - \lambda_N & \lambda_2^2 - \lambda_N^2 & \dots & \lambda_2^{N-1} - \lambda_N^{N-1} \\ 0 & \lambda_3 - \lambda_N & \lambda_3^2 - \lambda_N^2 & \dots & \lambda_3^{N-1} - \lambda_N^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{N-1} - \lambda_N & \lambda_{N-1}^2 - \lambda_N^2 & \dots & \lambda_{N-1}^{N-1} - \lambda_N^{N-1} \\ 1 & \lambda_N & \lambda_N^2 & \dots & \lambda_N^{N-1} \end{pmatrix}$$

$$= (-1)^{N-1} \det\begin{pmatrix} \lambda_1 - \lambda_N & \lambda_1^2 - \lambda_N^2 & \dots & \lambda_1^{N-1} - \lambda_N^{N-1} \\ \lambda_2 - \lambda_N & \lambda_2^2 - \lambda_N^2 & \dots & \lambda_2^{N-1} - \lambda_N^{N-1} \\ \lambda_3 - \lambda_N & \lambda_3^2 - \lambda_N^2 & \dots & \lambda_3^{N-1} - \lambda_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N-1} - \lambda_N & \lambda_{N-1}^2 - \lambda_N^2 & \dots & \lambda_{N-1}^{N-1} - \lambda_N^{N-1} \end{pmatrix}. \tag{2.1.4}$$

We will label this resulting matrix in (2.1.4) as A. Next, we will appeal to the following identity to factorise the rows of A,

$$x^j - y^j = (x-y) \sum_{k=0}^{j-1} x^k y^{j-1-k}\,. \tag{2.1.5}$$

This means we can write $A = \{a_{i,j}\}_{i,j=1,\ldots,N-1}$ where

$$a_{i,j} = (\lambda_i - \lambda_N)\sum_{k=0}^{j-1} \lambda_i^k \lambda_N^{j-1-k} \,. \tag{2.1.6}$$

Hence, we can now factorise out the $(\lambda_i - \lambda_N)$ entry from each row, resulting in

$$A = \begin{pmatrix} \lambda_1 - \lambda_N & 0 & \ldots & 0 \\ 0 & \lambda_2 - \lambda_N & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{N-1} - \lambda_N \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 + \lambda_N & \ldots & \sum_{k=0}^{N-2} \lambda_1^k \lambda_N^{(N-2)-k} \\ 1 & \lambda_2 + \lambda_N & \ldots & \sum_{k=0}^{N-2} \lambda_2^k \lambda_N^{(N-2)-k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N-1} + \lambda_N & \ldots & \sum_{k=0}^{N-2} \lambda_{N-1}^k \lambda_N^{(N-2)-k} \end{pmatrix} . \tag{2.1.7}$$

We will label the right hand matrix as $B = \{b_{i,j}\}_{i,j=1,\ldots,N-1}$, where after switching the order of exponents (due to the symmetry) for notational convenience, we have

$$\begin{aligned} b_{i,j} = \sum_{k=0}^{j-1} \lambda_i^{j-1-k} \lambda_N^k &= \lambda_i^{j-1} + \sum_{k=1}^{j-1} \lambda_i^{j-1-k} \lambda_N^k \\ &= \lambda_i^{j-1} + \lambda_N \sum_{k=1}^{j-1} \lambda_i^{j-1-k} \lambda_N^{k-1} \\ &= \lambda_i^{j-1} + \lambda_N \sum_{k=0}^{(j-1)-1} \lambda_i^{(j-1)-1-k} \lambda_N^k \\ &= \lambda_i^{j-1} + \lambda_N b_{i,(j-1)} \,. \end{aligned} \tag{2.1.8}$$

Clearly this looks very close to our desired Vandermonde matrix, and indeed due to this recursive definition in (2.1.8) we can subtract λ_N times the $(j-1)$th column from jth column, to recover our Vandermonde matrix. This series of operations corresponds to right multiplication by an upper triangular matrix with 1 on the diagonal and $-\lambda_N$ on the upper diagonal. In other words, we have now factorised

$$A = \begin{pmatrix} \lambda_1 - \lambda_N & 0 & \ldots & 0 \\ 0 & \lambda_2 - \lambda_N & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{N-1} - \lambda_N \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N-1} & \ldots & \lambda_{N-1}^{N-1} \end{pmatrix} \begin{pmatrix} 1 & -\lambda_N & 0 & \ldots & 0 \\ 0 & 1 & -\lambda_N & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda_N \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} . \tag{2.1.9}$$

Hence, taking the determinant of A and noting that that the determinant of the upper triangular matrix is clearly 1, we can simplify (2.1.4) using our inductive hypothesis:

$$\begin{aligned} \det[\lambda_a^{b-1}]_{a,b=1,\ldots,N} &= (-1)^{N-1} \left(\prod_{l=1}^{N-1} (\lambda_l - \lambda_N) \right) \det[\lambda_a^{b-1}]_{a,b=1,\ldots,N-1} \\ &= \prod_{l=1}^{N-1} (\lambda_N - \lambda_l) \prod_{1 \le l < m \le N-1} (\lambda_m - \lambda_l) = \prod_{1 \le l < m \le N} (\lambda_m - \lambda_l) \,. \end{aligned} \tag{2.1.10}$$

Thus by the principal of mathematical induction, the Vandermonde determinant is indeed a determinant. □

2.2) Change of measure for USp($2N$)

Consider the Cartain pair $\mathfrak{h}\oplus\mathfrak{p}$ where $\mathfrak{h} = \text{ASelf}(2N)$ are the anti-Hermitian anti-self-dual matrices

$$\text{ASelf}(2N) = \{A = -A^\dagger = -\hat{\tau}_2 A^T \hat{\tau}_2 \in \text{gl}_{\mathbb{C}}(2N)\}\,, \tag{2.2.1}$$

and $\mathfrak{p} = \text{Self}(2N)$ are the Hermitian-self-dual matrices

$$\text{Self}(2N) = \{A = A^\dagger = \hat{\tau}_2 A^T \hat{\tau}_2 \in \text{gl}_{\mathbb{C}}(2N)\}\,. \tag{2.2.2}$$

Hence, the invariance group is the unitary symplectic group USp($2N$). We will derive the change of Lebesgue measure for the two symmetric matrix spaces.

Firstly, we define two Abelian subalgebras

$$\text{For } \mathfrak{h} : \mathfrak{a}_{\mathfrak{h}} = \{\text{diag}(i\lambda_1\tau_3, \ldots, i\lambda_N\tau_3) \mid \lambda_j \in \mathbb{R}\}\,, \tag{2.2.3}$$
$$\text{For } \mathfrak{p} : \mathfrak{a}_{\mathfrak{p}} = \{\text{diag}(\lambda_1 \mathbb{1}_2, \ldots, \lambda_N \mathbb{1}_2) \mid \lambda_j \in \mathbb{R}\}\,, \tag{2.2.4}$$

where $i\tau_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Clearly these are subalgebras, and since the elements are just diagonal matrices and they clearly commute and so they are necessarily Abelian subalgebras. We can then check for maximilatiy.

Firstly selecting a fixed $X \in \text{ASelf}(2N)$, we will assume $[\Lambda, X]_- = 0$ holds for all $\Lambda \in \mathfrak{a}_{\mathfrak{h}}$, which we can write explicitly in 2x2 block form, where $\lambda_j, \lambda_k \in \mathbb{R}$, $z_{jk}^{(m)} \in \mathbb{C}$ and $j, k = 1, \ldots, N$,

$$\begin{aligned} 0 = \{[\Lambda, X]_-\}_{j,k} &= \begin{pmatrix} \lambda_j i & 0 \\ 0 & -\lambda_j i \end{pmatrix} \begin{pmatrix} z_{jk}^{(1)} & z_{jk}^{(2)} \\ z_{jk}^{(3)} & z_{jk}^{(4)} \end{pmatrix} - \begin{pmatrix} z_{jk}^{(1)} & z_{jk}^{(2)} \\ z_{jk}^{(3)} & z_{jk}^{(4)} \end{pmatrix} \begin{pmatrix} \lambda_k i & 0 \\ 0 & -\lambda_k i \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_j - \lambda_k) i z_{jk}^{(1)} & (\lambda_j + \lambda_k) i z_{jk}^{(2)} \\ -(\lambda_j + \lambda_k) i z_{jk}^{(3)} & -(\lambda_j - \lambda_k) i z_{jk}^{(4)} \end{pmatrix}\,. \end{aligned} \tag{2.2.5}$$

If $j = k$, then we necessarily have for all $\lambda_j \in \mathbb{R}$,

$$\begin{pmatrix} 0 & 2\lambda_j i z_{jj}^{(2)} \\ -2\lambda_j i z_{jj}^{(3)} & 0 \end{pmatrix} = 0\,, \quad \text{so } X_{j,j} = \begin{pmatrix} z_{jj}^{(1)} & 0 \\ 0 & z_{jj}^{(4)} \end{pmatrix}\,, \tag{2.2.6}$$

and then imposing $X + X^\dagger = 0$ and $X + \hat{\tau}_2 X^T \hat{\tau}_2 = 0$ yields

$$X_{j,j} = \begin{pmatrix} x_{jj} i & 0 \\ 0 & -x_{jj} i \end{pmatrix}\,, \quad \text{where } x_j \in \mathbb{R}\,. \tag{2.2.7}$$

If $j \neq k$ then from (2.2.5) we see that $z_{j,k}^{(1)} = z_{j,k}^{(2)} = z_{j,k}^{(3)} = z_{j,k}^{(4)} = 0$ because it must be true for all $\lambda_j, \lambda_k \in \mathbb{R}$. Therefore, if X is in ASelf($2N$) and commutes with all $\Lambda \in \mathfrak{a}_{\mathfrak{h}}$ then

$$X = \text{diag}(x_1 i\tau_3, \ldots, x_N i\tau_3) \in \mathfrak{a}_{\mathfrak{h}} \quad \text{where } x_j \in \mathbb{R} \tag{2.2.8}$$

and so $\mathfrak{a}_{\mathfrak{h}}$ is maximal.

We can then apply the same procedure for $Y \in \mathrm{Self}(2N)$ and $\Lambda \in \mathfrak{a}_\mathfrak{p}$ where we compute for the same conditions (and $w_{jk}^{(m)} \in \mathbb{C}$)

$$\begin{aligned} 0 = \{[\Lambda, Y]_-\}_{j,k} &= \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix} \begin{pmatrix} w_{jk}^{(1)} & w_{jk}^{(2)} \\ w_{jk}^{(3)} & w_{jk}^{(4)} \end{pmatrix} - \begin{pmatrix} w_{jk}^{(1)} & w_{jk}^{(2)} \\ w_{jk}^{(3)} & w_{jk}^{(4)} \end{pmatrix} \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_j - \lambda_k) w_{jk}^{(1)} & (\lambda_j - \lambda_k) w_{jk}^{(2)} \\ (\lambda_j - \lambda_k) w_{jk}^{(3)} & (\lambda_j - \lambda_k) w_{jk}^{(4)} \end{pmatrix} . \end{aligned} \tag{2.2.9}$$

If $j \neq k$ then all terms $w_{j,k}^{(m)}$ vanish again since the statement is true for all $\lambda_j, \lambda_k \in \mathbb{R}$. For $j = k$, we can apply the Hermiticity to see that

$$Y_{j,j} = \begin{pmatrix} x_{j,j}^{(1)} & x_{j,j}^{(2)} + y_{j,j}^{(2)} i \\ x_{j,j}^{(2)} - y_{j,j}^{(2)} i & x_{j,j}^{(4)} \end{pmatrix} \quad \text{where } x_{j,k}^{(m)}, y_{j,k}^{(m)} \in \mathbb{R} , \tag{2.2.10}$$

and then applying $Y - \hat{\tau}_2 Y^T \hat{\tau}_2 = 0$ we see

$$0 = Y - \hat{\tau}_2 Y^T \hat{\tau}_2 = \begin{pmatrix} (x_{j,j}^{(1)} - x_{j,j}^{(4)}) & 2(x_{j,j}^{(2)} + y_{j,j}^{(2)} i) \\ 2(x_{j,j}^{(2)} + y_{j,j}^{(2)} i) & x_{j,j}^{(4)} - x_{j,j}^{(1)} \end{pmatrix} , \tag{2.2.11}$$

so $x_{j,j}^{(1)} = x_{j,j}^{(4)}$ and $x_{j,j}^{(2)} = y_{j,j}^{(2)} = 0$ so we conclude that

$$Y = \mathrm{diag}(x_1 \mathbb{1}_2, \ldots, x_N \mathbb{1}_2) \in \mathfrak{a}_\mathfrak{p} \quad \text{where } x_j \in \mathbb{R} . \tag{2.2.12}$$

Therefore, both $\mathfrak{a}_\mathfrak{h}$ and $\mathfrak{a}_\mathfrak{p}$ are maximal Abelian subalgebras for $\mathfrak{h}$ and $\mathfrak{p}$. Then, since the unitary symplectic group is compact, we know that any other maximal Abelian subalgebra is equivalent to $\mathfrak{a}_\mathfrak{h}$ under a similarity transformation. Hence, these choices are unique enough to say that $\mathfrak{a}_\mathfrak{h}$ and $\mathfrak{a}_\mathfrak{p}$ are *the* maximal Abelian subalgebras for $\mathfrak{h}$ and $\mathfrak{p}$. Further, the orbit agrees with the whole matrix space $\mathrm{USp}(2N)$ because it is compact and $\mathfrak{a}$ is a maximal Abelian subspace.

We next proceed to calculating the normalisers of the two maximal Abelian subalgebras, $\mathcal{N}_{\mathrm{USp}(2N)}(\mathfrak{a}_\mathfrak{h})$ for $\mathfrak{h} = \mathrm{ASelf}(2N)$ and $\mathcal{N}_{\mathrm{USp}(2N)}(\mathfrak{a}_\mathfrak{p})$ for $\mathfrak{p} = \mathrm{Self}(2N)$. For the first case, this is the set of $U \in \mathcal{N}_{\mathrm{USp}(2N)}(\mathfrak{a}_\mathfrak{h})$ such that

$$U \Lambda U^{-1} = \tilde{\Lambda} \quad \text{for all } \Lambda \in \mathfrak{a}_\mathfrak{h} \, , \text{where } \tilde{\Lambda} \in \mathfrak{a}_\mathfrak{h} \, . \tag{2.2.13}$$

We can see that these two objects have the same eigenvalues by comparing their characteristic equations,

$$\det(\tilde{\Lambda} - x\mathbb{1}) = \det(U \Lambda U^{-1} - x\mathbb{1}) = \det(U) \det(\Lambda - U^{-1} x \mathbb{1} U) \det(U^{-1}) = \det(\Lambda - x\mathbb{1}) . \tag{2.2.14}$$

Thus, $\mathrm{diag}(\tilde{\lambda_1} i, -\tilde{\lambda_1} i, \ldots, \tilde{\lambda_N} i, -\tilde{\lambda_N} i)$ has to be a permutation of $\mathrm{diag}(\lambda_1 i, -\lambda_1 i, \ldots, \lambda_N i, -\lambda_N i)$. We see that we can switch $\lambda_j \leftrightarrow -\lambda_j$, which corresponds to $\mathbb{Z}_2$ and since we can do N of these permutations, this gives us $\mathbb{Z}_2^N$ in the normaliser. Also, we can permute all of the λ_j in different ways which correspond to the symmetric group $\mathbb{S}_N$, so the finite part of our group is $\mathbb{Z}_2^N \times \mathbb{S}_N$.

To find the operators U' for which all Λ are purely eigenvectors, we choose a $U' \in \mathbb{Z}_2^N \times \mathbb{S}_N \subset \mathrm{USp}(2N)$, and then using the same derivation in the lecture we must have $[U'^{-1}U, \Lambda]_- = 0$. We could then perform the same calculation as in step 1, which we omit for brevity here, to get that $U'U^{-1} = \mathrm{diag}(U_1, \dots, U_N)$ where $U_j = \cos(\psi_j) + \sin(\psi_j)\tau_2 = e^{i\psi_j} \in \mathrm{U}(1)$, where $\mathrm{U}(1)$ represents the one dimensional complex numbers with magnitude 1. Hence putting all of this together we get that $\mathcal{N}_{\mathrm{USp}(2N)}(\mathfrak{a}_\mathfrak{h}) = \mathrm{U}^N(1) \times \mathbb{Z}_2^N \times \mathbb{S}_N$. When taking this to the Lie Algebra level, we only care about the non-finite part $\mathrm{U}^N(1)$, whose corresponding Lie Algebra is the anti-Hermitian matrices of degree 1, which corresponds to being purely imaginary. Representing complex numbers in a 2×2 structure, this is equivalent to being real anti-Hermitian of degree 2. Therefore, we take the quotient $\mathfrak{h}/\mathfrak{n}_\mathcal{H}(\mathfrak{a}_\mathfrak{h}) = \mathrm{ASelf}(2N)/\mathrm{AHerm}_\mathbb{R}^N(2)$.

For $\mathfrak{p} = \mathrm{Self}(2N)$, we use the same argument to find the permutations of $\mathrm{diag}(\lambda_1, \lambda_1, \dots, \lambda_N, \lambda_N)$. Clearly this time we only have the symmetric permutations, giving us $\mathbb{S}_N$ for the finite group. Again when performing the next procedure, we would arrive at a quarternion-esque structure resembling SU(2), which in our realm is roughly equivalent to USp(2). Hence we have $\mathcal{N}_{\mathrm{USp}(2N)}(\mathfrak{a}_\mathfrak{p}) = \mathrm{USp}^N(2) \times \mathbb{S}_N$. On the Lie Algebra level, $\mathrm{USp}^N(2)$ corresponds to complex matrices X that anti-commute with $\hat{\tau}_2 = \tau_2 \otimes \mathbb{1}_N$, $[\hat{\tau}_2, X]_+ = 0$, that is, the quaternion anti-Hermitian matrices. I don't really know how to denote these but we can just say $\mathfrak{h}/\mathfrak{n}_\mathcal{H}(\mathfrak{a}_\mathfrak{p}) = \mathrm{Self}(2N)/\mathrm{AHerm}^N(2)$.

Similar to the lecture notes, this quotienting will give us nearly-0 diagonal entries for an element A in either of these quotient spaces. However, the diagonals are not quite 0 as there is still some symmetry preserved from our two normalisers. In taking this quotient, we are effectively removing the zero roots in the end, putting the degeneracy in the bin. Ultimately this ensures that we don't have anything to analyse for the $j = k$ block of a matrix A in the quotient spaces.

Returning to $\mathfrak{h}$, we can analyse the 2×2 block $\{[\Lambda, A]\}_{j,k}$ for $j \neq k$ that results from taking the commutator $[\Lambda, A]$ for all $\Lambda \in \mathfrak{a}$ and a fixed A in the quotient space. This is precisely what we have calculated in (2.2.5), which we can now write as a matrix over $(A_{jk}^{(1)}, A_{jk}^{(2)}, A_{jk}^{(3)}, A_{jk}^{(4)}) \in \mathbb{R}^4$, namely

$$\begin{pmatrix} i(\lambda_j - \lambda_k) & 0 & 0 & 0 \\ 0 & i(\lambda_j + \lambda_k) & 0 & 0 \\ 0 & 0 & -i(\lambda_j + \lambda_k) & 0 \\ 0 & 0 & 0 & -i(\lambda_j - \lambda_k) \end{pmatrix}. \tag{2.2.15}$$

The eigvenvalues of this matrix are simply the diagonal entries, hence allowing us to write for the roots the same as in the lecture notes

$$\mathcal{R} = \{\Lambda \mapsto i(L_1\lambda_j + L_2\lambda_k) \big| 1 \leq j < k \leq N,\ L_1, L_2 = \pm 1\}. \tag{2.2.16}$$

Hence we can now write

$$\begin{aligned} \prod_{\alpha \in \mathcal{R}} |\alpha(\Lambda)| &= \prod_{1 \leq j < k \leq N} |(\lambda_j - \lambda_k)(\lambda_j + \lambda_k)(-\lambda_j + \lambda_k)(-\lambda_j - \lambda_k)| \\ &= \prod_{1 \leq j < k \leq N} |(\lambda_j^2 - \lambda_k^2)|^2 = |\Delta_N(\Lambda^2)|^2. \end{aligned} \tag{2.2.17}$$

Knowing that we didn't have to take the normalisation constant, in the end this yields for $\mathfrak{h}$

$$d[X] = C_l |\Delta_N(\Lambda^2)|^2 d[\Lambda] d\mu(U)\,. \tag{2.2.18}$$

For $\mathfrak{p}$, performing an identical calculation from (2.2.9) gives us

$$\prod_{\alpha \in \mathcal{R}} |\alpha(\Lambda)| = \prod_{1 \le j < k \le N} |(\lambda_j - \lambda_k)(\lambda_j - \lambda_k)(\lambda_j - \lambda_k)(\lambda_j - \lambda_k)| = |\Delta_N(\Lambda)|^4\,. \tag{2.2.19}$$

Hence, for this we have

$$d[X] = C_l |\Delta_N(\Lambda)|^4 d[\Lambda] d\mu(U)\,. \tag{2.2.20}$$

We note that there was clearly some fudging in understanding the quotient space analysis on the previous page, but ultimately this does appear to agree with the calculation of the general jpdf with $\beta = 2, 4$ respectively. Thus we have calculated the change of measure for the two matrix spaces $\mathfrak{h} = \mathrm{ASelf}(2N)$ and $\mathfrak{p} = \mathrm{Self}(2N)$ that arise from the Cartan decomposition.

Q3. Monic orthogonal polynomials

We will first prove the three-term recurrence relation. Let $\{q_j\}_{j=0,\ldots,N}$ be a monic orthogonal basis of $\mathbb{P}_{N+1}$ with respect to a weight w and normalisations h_j. Then we will show these polynomials obey the recurrence relation

$$xq_j(x) = q_{j+1}(x) + d_j q_j(x) + e_j q_{j-1}(x)\,. \tag{3.0.1}$$

It is easy to see that for the monomial x we have, for suitable (i.e. regular enough) real functions f and g we have

$$\langle xf|g\rangle_w = \int_{\infty}^{\infty} (xf(x))\,g(x)\,w(x)dx = \int_{\infty}^{\infty} f(x)\,(xg(x))\,w(x) = \langle f|xg\rangle_w\,. \tag{3.0.2}$$

Since $xq_j(x) \in \mathbb{P}_{j+1}$ and is clearly monic, we can write

$$xq_j(x) = \sum_{k=0}^{j+1} a_{j,k} q_k(x) \quad \text{where } a_{j,k} \in \mathbb{R}\,. \tag{3.0.3}$$

Since $xq_j(x)$ is monic, we see straight away that $a_{j,j+1} = 1$. Multiplying both sides by q_m for a fixed $0 \leq m \leq j+1$ and integrating we see

$$\begin{aligned}
\langle xq_j|q_m\rangle_w = \int_{-\infty}^{\infty} xq_j(x)q_m(x)w(x)dx &= \sum_{k=0}^{j+1} a_{j,k} \int_{-\infty}^{\infty} q_k(x)q_m(x)w(x)dx &\quad (3.0.4)\\
&= \sum_{k=0}^{j+1} a_{j,k} h_m \delta_{km} = a_{j,m} h_m\,, &\quad (3.0.5)
\end{aligned}$$

which tells us that (where $h_m = \langle q_m|q_m\rangle$ is the normalisation constant)

$$a_{j,m} = \frac{\langle xq_j|q_m\rangle_w}{h_m} = \frac{\langle q_j|xq_m\rangle_w}{h_m}\,. \tag{3.0.6}$$

But since xq_m is a monic polynomial of degree $m+1$, we know that xq_m must be orthogonal to q_j (i.e. $\langle q_j|xq_m\rangle_w = 0$) for $0 \leq m \leq j-1$. So in returning to (3.0.3) we now have

$$xq_j(x) = q_{j+1}(x) + \frac{\langle xq_j|q_j\rangle_w}{h_j} q_j(x) + \frac{h_j}{h_{j-1}} q_{j-1}(x)\,, \tag{3.0.7}$$

thus proving the necessary recurrence relation. For the Christoffel-Darboux formula, we consider (where we have $e_j = h_j/h_{j-1}$ and use the recurrence relation in the second line)

$$\begin{aligned}
&(x_1 - x_2)\sum_{j=0}^{N-1} \frac{q_j(x_1)q_j(x_2)}{h_j} = \sum_{j=0}^{N-1} \left\{\frac{x_1 q_j(x_1)q_j(x_2)}{h_j}\right\} - \sum_{j=0}^{N-1} \left\{\frac{q_j(x_1)x_2 q_j(x_2)}{h_j}\right\}\\
&= \sum_{j=0}^{N-1} \left\{\frac{[q_{j+1}(x_1) + d_j q_j(x_1) + e_j q_{j-1}(x_1)]\,q_j(x_2)}{h_j}\right\} - \sum_{j=0}^{N-1} \left\{\frac{q_j(x_1)\,[q_{j+1}(x_2) + d_j q_j(x_2) + e_j q_{j-1}(x_2)]}{h_j}\right\}\\
&= \sum_{j=0}^{N-1} \left\{\frac{q_{j+1}(x_1)q_j(x_2) - q_j(x_1)q_{j+1}(x_2)}{h_j}\right\} - \sum_{j=0}^{N-1} \left\{\frac{q_j(x_1)q_{j-1}(x_2) - q_{j-1}(x_1)q_j(x_2)}{h_{j-1}}\right\}\\
&= \frac{q_N(x_1)q_{N-1}(x_2) - q_{N-1}(x_1)q_N(x_2)}{h_{N-1}}\,. \qquad (3.0.8)
\end{aligned}$$

To get to the final line we implicitly made a substitution in the first sum of $j' = j+1$, forcing most terms to cancel. Thus dividing by $(x_1 - x_2)$ yields the formula. $\square$

Q4. Local spectral statistics of GUE

We will derive the local spectral statistics of the GUE with distribution

$$P(X) = \frac{\exp[-N\mathrm{tr}X^2/2]}{(2\pi/N)^{N/2}(\pi/N)^{N(N-1)/2}}\,, \quad X \in \mathrm{Herm}(N)\,. \tag{4.0.1}$$

We start by considering the double contour formula for the kernel

$$\begin{aligned}\sqrt{N}K_N^{(GUE)}(\sqrt{N}x_1, \sqrt{N}x_2) = \frac{N!}{\sqrt{2\pi}N^{N-1/2}} \frac{\exp[-Nx_2^2/2]}{x_1 - x_2} \\ \times \oint \frac{dz_1 dz_2}{(2\pi i)^2} \frac{z_2 - z_1}{z_1^{N+1} z_2^{N+1}} \exp\left[-N\left(\frac{z_1^2 + z_2^2}{2} - x_1 z_1 - x_2 z_2\right)\right]\,.\end{aligned} \tag{4.0.2}$$

Collecting terms in the double contour, we can express the integrand as

$$\oint dz_1 dz_2 g(z_1, z_2) \exp[-N(f(z_1) + f(z_2))]$$

$$\text{where} \quad g(z_1, z_2) = \frac{z_2 - z_1}{z_2 z_1} \qquad \text{and} \qquad f(z_j) = \frac{1}{2}z_j^2 - x_j z_j + \log z_j\,. \tag{4.0.3}$$

Since we are 'zooming in' on x_0 we let the spectral variables $x_1 = x_2 = x_0$ to find the saddle points, and hence we can calculate

$$f'(z) = z - x_0 + \frac{1}{z} = 0\,, \qquad \text{so} \quad z_\pm = \frac{1}{2}\left(x_0 \pm \sqrt{x_0^2 - 4}\right) = x_0/2 \pm i\sqrt{1 - x_0^2/4}\,.$$

We note here that $|z_\pm| = 1$ so we don't have to rescale our contour to ensure that it goes through the saddle points. We notice too that the second derivative is

$$f''(z) = 1 - \frac{1}{z^2}\,, \quad \text{which yields for} \tag{4.0.4}$$

$$\underline{x_0 = 0}: \; f''(z_\pm) = 1 - \frac{1}{(\pm i)^2} = 2 \qquad \text{and} \qquad \underline{x_0 = 2}: \; f''(z_\pm) = 1 - \frac{1}{1^2} = 0\,. \tag{4.0.5}$$

Like in the example in the lecture notes, we have two integration variables with the same action, leading to four saddle points $(z_1, z_2) = (z_\pm, z_\pm)$ and $(z_1, z_2) = (z_\pm, z_\mp)$. But once again, the $z_2 - z_1$ factor in the pre-exponential removes this first pair, hence we are only interested in the second pair (and we will see at the soft edge there is only one saddle point anyway). This second pair will give us the expansion $z_2 - z_1 = \mp 2i\sqrt{1 - x_0^2/4}$.

4.1) Bulk scaling

We start by analysing the non-degenerate bulk point $x_0 = 0$. We note that here $z_\pm = \pm i$. We can rescale our variables:

$$(z_1, z_2) = (z_\pm, z_\mp) + \frac{1}{\sqrt{2N}}(\delta z_1, \delta z_2)\,, \tag{4.1.1}$$

$$\text{with measure} \qquad \frac{dz_1 dz_2}{(2\pi i)^2} = \frac{d\delta z_1 d\delta z_2}{2N(2\pi i)^2} \tag{4.1.2}$$

and the pre-exponential factor becomes

$$\frac{z_2 - z_1}{z_1 z_2} \approx \frac{z_\mp - z_\pm}{z_\pm z_\mp} = \mp 2i\,. \tag{4.1.3}$$

We next look to rescale our spectral variables x_1 and x_2, which we can do by employing the useful approximation from the lecture notes $\lambda_j = \lambda_0 + \bar{s}(\lambda_0)\delta\lambda_j$. In our case the mean level density for the GUE is $\bar{\rho}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$, and using $\bar{s}(\lambda_0) \approx 1/(N\bar{\rho}(\lambda_0))$ this rescaling yields

$$x_j = x_0 + \frac{2\pi}{N\sqrt{4-x_0^2}}\delta x_j = \frac{\pi}{N}\delta x_j\,. \tag{4.1.4}$$

Before performing the integral approximation we can evaluate the kernel after the spectral variable rescaling:

$$\begin{aligned}\frac{\pi\sqrt{N}}{N}K_N^{(GUE)}\left(\sqrt{N}\left(\frac{\pi}{N}\delta x_1\right), \sqrt{N}\left(\frac{\pi}{N}\delta x_2\right)\right) &= \frac{\pi}{N}\frac{N!}{\sqrt{2\pi}N^{N-1/2}}\frac{\exp[-\frac{\pi^2}{2N}(\delta x_2)^2]}{\frac{\pi}{N}(\delta x_1 - \delta x_2)} \\ \times \oint \frac{dz_1 dz_2}{(2\pi i)^2}\frac{z_2 - z_1}{z_1^{N+1}z_2^{N+1}}\exp\left[-N\left(\frac{z_1^2+z_2^2}{2} - \frac{\pi}{N}\delta x_1 z_1 - \frac{\pi}{N}\delta x_2 z_2\right)\right]&. \end{aligned}\tag{4.1.5}$$

We can then use Stirling's approximation to write

$$\frac{N!}{\sqrt{2\pi}N^{N-1/2}} \approx Ne^{-N} \quad \text{for } N \gg 1\,, \tag{4.1.6}$$

$$\text{and note that} \quad \exp\left[-\frac{\pi^2}{2N}(\delta x_2)^2\right] \to 1 \quad \text{for } N \gg 1 \text{ and } \delta x_2 \ll 1\,. \tag{4.1.7}$$

Hence, we can then rewrite our integral as

$$\begin{aligned}\frac{\pi\sqrt{N}}{N}K_N^{(GUE)}\left(\sqrt{N}\left(\frac{\pi}{N}\delta x_1\right), \sqrt{N}\left(\frac{\pi}{N}\delta x_2\right)\right) &= \frac{Ne^{-N}}{\delta x_1 - \delta x_2} \\ \times \oint \frac{dz_1 dz_2}{(2\pi i)^2}\frac{z_2 - z_1}{z_1 z_2}\exp[\pi\delta x_1 z_1 + \pi\delta x_2 z_2]\exp\left[-N\left(\frac{z_1^2+z_2^2}{2} + \log z_1 + \log z_2\right)\right]&. \end{aligned}\tag{4.1.8}$$

We note that we have an additional pre-factor term so we calculate, letting $(z_1, z_2) \approx (z_\pm, z_\mp)$,

$$\exp[\pi\delta x_1 z_1 + \pi\delta x_2 z_2] \approx \exp\big[\pi\delta x_1(\pm i) + \pi\delta x_2(\mp i)\big] = \exp[\pm i\pi(\delta x_1 - \delta x_2)]\,. \tag{4.1.9}$$

We now have all of the ingredients to perform the saddle point approximation. We can now rewrite (4.1.8) as

$$\begin{aligned}\frac{\pi\sqrt{N}}{N}K_N^{(GUE)}\left(\sqrt{N}\left(\frac{\pi}{N}\delta x_1\right), \sqrt{N}\left(\frac{\pi}{N}\delta x_2\right)\right) &= \frac{Ne^{-N}}{\delta x_1 - \delta x_2}(\mp 2i)\exp[\pm i\pi(\delta x_1 - \delta x_2)] \\ \times \int_{\mathbb{R}^2}\frac{d\delta z_1 d\delta z_2}{2N(2\pi i)^2}\exp\left[-N\left(\frac{(\pm i)^2 + (\mp i)^2}{2}\right) + \log(\pm i) + \log(\mp i) - \frac{\delta z_1^2}{2} - \frac{\delta z_2^2}{2}\right]& \\ = \frac{Ne^{-N}}{\delta x_1 - \delta x_2}\frac{(\mp 2i)e^N}{2N(2\pi i)^2}\exp[\pm i\pi(\delta x_1 - \delta x_2)]\int_{\mathbb{R}^2} d\delta z_1\, d\delta z_2\, \exp\left[-\frac{\delta z_1^2}{2} - \frac{\delta z_2^2}{2}\right]& \\ = \frac{1}{(2\pi)^2}\frac{2\sin(\pi(\delta x_1 - \delta x_2))}{\delta x_1 - \delta x_2}(2\pi) = \frac{\sin(\pi(\delta x_1 - \delta x_2))}{\pi(\delta x_1 - \delta x_2)} = K_{\text{sine}}(\delta x_1, \delta x_2)&\,. \end{aligned}$$

In the third line (equals sign) we proceeded to add the contributions from the two saddle points, hence giving us the $\sin(\pi(\delta x_1 - \delta x_2))$ from hypoerbolic trig identities. We also picked up a negative sign due to the direction of the contour at the two saddle points being different. But all in all, we arrived at the famous sine kernel! Yay for us *pats back*.

4.2) Soft edge scaling

We can now perform a similar procedure for the *degenerate* soft edge $x_0 = 2$. We note that our saddle point expansion will have to be to higher order due to this degeneracy, $f''(z_\pm) = 0$. We note here that we will relabel $z_\pm = z_0 = 1$ since $z_+ = z_-$ in this calculation, so there is only one saddle point $(z_1, z_2) = (1, 1)$ - this will have important effects later on. Hence we calculate:

$$f'''(z) = \frac{2}{z^3}\,, \qquad \text{so} \quad f'''(z_0) = \frac{2}{1^3} = 2 \tag{4.2.1}$$

and so we rescale the integration variables as

$$(z_1, z_2) = (z_0, z_0) + \frac{i}{(2N)^{1/3}}(\delta z_1, \delta z_2)\,, \tag{4.2.2}$$

$$\text{with measure} \qquad \frac{dz_1 dz_2}{(2\pi i)^2} = \frac{d\delta z_1 d\delta z_2}{(2N)^{2/3}(2\pi)^2} \tag{4.2.3}$$

The factor of i must be included in order to ensure our contour is going in the right direction through the saddle point. The fact that $z_1 = z_2$ at the saddle point means that we have to approximate the pre-N-exponential term by going one higher term in the Taylor expansion. So in letting $g(z) = 1/z$, hence $g(z_1) - g(z_2) = \frac{z_2 - z_1}{z_2 z_1}$, in taking appropriate derivatives and such we have

$$\frac{z_2 - z_1}{z_1 z_2} \approx i \frac{\delta z_2 - \delta z_1}{(2N)^{1/3}}\,. \tag{4.2.4}$$

Again, the i appears due to our parametrisation above. To rescale our spectral variables, this time we need to use a different approximation for the mean level spacing as our previous one vanishes at the soft edge $x_0 = 2$. We note that we can write

$$\bar{\rho}(\lambda) d\lambda = \frac{1}{2\pi}\sqrt{4 - \lambda^2} \approx \frac{1}{\pi}\sqrt{\delta\tilde{\lambda}}\, d\delta\tilde{\lambda}\,, \tag{4.2.5}$$

therefore the number of eigenvalues close to the soft edge is

$$N([2 - \delta\tilde{\lambda}, 2]) = N \int_{2-\delta\tilde{\lambda}}^{2} \bar{\rho}(\lambda) d\lambda = \frac{2N}{3\pi}\delta\tilde{\lambda}^{3/2}\,. \tag{4.2.6}$$

We then approximate the mean level spacing by letting $N([2 - \delta\tilde{\lambda}, 2]) \approx 1$. Discarding constants as $N \gg 1$, we get that at the soft edge we have

$$\bar{s}(2) = \frac{1}{N^{2/3}}\,, \tag{4.2.7}$$

so we parametrise the spectral variables as (where γ is a constant that we fix later)

$$x_j = 2 + \frac{\gamma}{N^{2/3}}\delta x_j \,. \tag{4.2.8}$$

We can now observe our first form of the kernel (where we use the same Stirling approximation as before)

$$\begin{aligned}
&\frac{\gamma\sqrt{N}}{N^{2/3}}K_N^{(GUE)}\left(\sqrt{N}\left(2+\frac{\gamma}{N^{2/3}}\delta x_1\right), \sqrt{N}\left(2+\frac{\gamma}{N^{2/3}}\delta x_2\right)\right) = \frac{\gamma N e^{-N}}{N^{2/3}}\frac{\exp[-N\left(2+\frac{\gamma}{N^{2/3}}\delta x_2\right)^2/2]}{\frac{\gamma}{N^{2/3}}(\delta x_1-\delta x_2)} \\
&\times \oint \frac{dz_1 dz_2}{(2\pi i)^2}\frac{z_2-z_1}{z_1^{N+1}z_2^{N+1}}\exp\left[-N\left(\frac{z_1^2+z_2^2}{2}-\left(2+\frac{\gamma}{N^{2/3}}\delta x_1\right)z_1-\left(2+\frac{\gamma}{N^{2/3}}\delta x_2\right)z_2\right)\right] \\
&= \frac{Ne^{-N}\exp[-2N-2\gamma N^{1/3}\delta x_2]}{\delta x_1-\delta x_2}\oint\frac{dz_1dz_2}{(2\pi i)^2}\frac{z_2-z_1}{z_1 z_2}\exp\left[\gamma N^{1/3}(\delta x_1 z_1+\delta x_2 z_2)\right] \\
&\times\exp\left[-N\left(\frac{z_1^2+z_2^2}{2}-2z_1-2z_2+\log z_1+\log z_2\right)\right]\,.
\end{aligned} \tag{4.2.9}$$

In the second line (equals sign) we let the pre-integral term $\exp[-\frac{\gamma^2}{2N^{1/3}}\delta x_2^2] \to 1$ in the limit. We can then calculate the leading order terms of the action at the saddle point,

$$-Nf\left(1+\frac{i}{(2N)^{1/3}}\delta z_j\right) \approx -Nf(1) + i\frac{\delta x_z^3}{3!} = \frac{3}{2}N + i\frac{\delta z_j^3}{3!}\,. \tag{4.2.10}$$

Our final ingredient is to expand the new exponential pre-factor term,

$$\exp\left[\gamma N^{1/3}(\delta x_1 z_1+\delta x_2 z_2)\right] \approx \exp\left[\gamma N^{1/3}(\delta x_1+\delta x_2) + i\frac{\gamma}{2^{1/3}}(\delta x_1\delta z_1+\delta x_2\delta z_2)\right]\,. \tag{4.2.11}$$

Therefore, our pasta has been on the boil and our sauce is comin right up. Mmm garlic.

$$\begin{aligned}
&\frac{\gamma\sqrt{N}}{N^{2/3}}K_N^{(GUE)}\left(\sqrt{N}\left(2+\frac{\gamma}{N^{2/3}}\delta x_1\right), \sqrt{N}\left(2+\frac{\gamma}{N^{2/3}}\delta x_2\right)\right) = \frac{Ne^{-3N}\exp[-2\gamma N^{1/3}\delta x_2]}{\delta x_1-\delta x_2} \\
&\times\oint\frac{d\delta z_1 d\delta z_2}{(2N)^{2/3}(2\pi)^2}\frac{i\delta z_2 - i\delta z_1}{(2N)^{1/3}}\exp\left[\gamma N^{1/3}(\delta x_1+\delta x_2)+i\frac{\gamma}{2^{1/3}}(\delta x_1\delta z_1+\delta x_2\delta z_2)\right] \\
&\times\exp\left[3N+\frac{i}{3!}(\delta z_1^3+\delta z_2^3)\right] \\
&= \frac{1}{2^3\pi^2}\frac{\exp[\gamma N^{1/3}(\delta x_1-\delta x_2)}{\delta x_1-\delta x_2}\oint d\delta z_1\, d\delta z_2\,(i\delta z_2-i\delta z_1)\exp\left[i\frac{\gamma}{2^{1/3}}(\delta x_1\delta z_1+\delta x_2\delta z_2)+\frac{i}{3!}(\delta z_1^3+\delta z_2^3)\right]\,,
\end{aligned}$$

and now we can rescale $\delta z_j \mapsto 2^{1/3}\delta z_j$ and $\gamma = 1$ to get

$$= \frac{1}{2^{7/3}\pi^2}\frac{\exp[N^{1/3}(\delta x_1-\delta x_2)}{\delta x_1-\delta x_2}\oint d\delta z_1\, d\delta z_2\,(i\delta z_2-i\delta z_1)\exp\left[i(\delta x_1\delta z_1+\delta x_2\delta z_2)+\frac{i}{3}(\delta z_1^3+\delta z_2^3)\right]\,. \tag{4.2.12}$$

Therefore, after so much blood sweat and tears holy heck would you look at the time its 1:30am, we get

$$\begin{aligned}
&\frac{2^{7/3}\pi^2\sqrt{N}}{N^{2/3}}\exp[-N^{1/3}(\delta x_1-\delta x_2)]K_N^{(GUE)}\left(\sqrt{N}\left(2+\frac{\gamma}{N^{2/3}}\delta x_1\right), \sqrt{N}\left(2+\frac{\gamma}{N^{2/3}}\delta x_2\right)\right) \\
&= K_{\text{Airy}}(\delta x_1, \delta x_2)\,.
\end{aligned} \tag{4.2.13}$$

Yay for the Airy kernel. Yay for kernels. Yay for random matrices. And most importantly, yay for bedtime. QED pasta. Yay for garlic. □