Partial Differential Equations Assignment 3

Liam Carroll - 830916

Due Date: 11th October 2021

Q1. Harmonic functions on the punctured disc

Let u be a harmonic function, so $\Delta u = 0$, on the punctured unit disc

$$\Omega_0 = \{x \in \mathbb{R}^2 \,|\, 0 < |x| < 1\}\,.$$

Part a)

Suppose u is continuous at the origin. Then we will show that u is harmonic across the unit disc $\Omega = \{x \in \mathbb{R}^2 \,|\, |x| < 1\}$. To do this we will show that u is weakly harmonic on Ω. Then by Theorem 15.1, this will imply that it is harmonic on all of Ω, without any need to redefine any points due to the continuity of u.

Let $\psi \in C_0^\infty(\Omega)$ be an arbitrary compactly supported smooth function. We want to show $\langle u, \Delta\psi\rangle = 0$. Since u is harmonic on Ω_0, it is weakly harmonic on Ω_0 by a simple application of integration by parts. So, for any $\psi_0 \in C_0^\infty(\Omega_0)$ we have $\langle u, \Delta\psi_0\rangle = 0$. This suggests our task becomes finding a function $F_\epsilon(|x|)$ such that:

1. $\psi(1 - F_\epsilon) \in C_0^\infty(\Omega_0)$ and;
2. $\lim_{\epsilon\to 0} \Delta(\psi F_\epsilon) = 0$,

to then write

$$\langle u, \Delta\psi\rangle = \langle u, \Delta(\psi(1 - f_\epsilon))\rangle + \langle u, \Delta(\psi f_\epsilon)\rangle\,.$$

If we find such a function, then the first term will be 0 since u is weakly harmonic, and the second term will go to 0 in the limit, implying $\langle u, \Delta\psi\rangle = 0$.

Let us craft a function that is 1 locally near the origin and harmonic outside. Recall that the fundamental radial solution to the Laplace equation is $k \log\|x\|$. This suggests we might write

$$f_\epsilon(x) = \begin{cases} 1 & \text{if } \|x\| < e^{-\frac{1}{\epsilon}} \\ -\epsilon\log\|x\| & \text{if } \|x\| > e^{-\frac{1}{\epsilon}} \end{cases}\,.$$

We see that $f_\epsilon(x)$ is continuous, but it is not differentiable, let alone smooth, at $\|x\| = e^{-\frac{1}{\epsilon}}$. This suggests we should mollify $f_\epsilon(x)$. Let $\delta < \min(e^{-\frac{1}{\epsilon}}, 1 - e^{-\frac{1}{\epsilon}})$, then we may define

$$F_{\epsilon,\delta}(x) = (f_\epsilon * K_\delta)(x)$$

where $K_\delta(x)$ is the approximation to the identity from Lecture 13. Due to our bound on δ, we have that

$$\Delta F_{\epsilon,\delta}(x) = \begin{cases} 0 & \text{if} \quad x < e^{-\frac{1}{\epsilon}} - \delta \\ \Delta \int_{\mathbb{R}^d} f_\epsilon(y) K_\delta(x-y)\,\mathrm{d}x & \text{if} \quad e^{-\frac{1}{\epsilon}} - \delta < x < e^{-\frac{1}{\epsilon}} + \delta \\ 0 & \text{if} \quad e^{-\frac{1}{\epsilon}} + \delta < x < 1 \end{cases}.$$

Unfortunately, I wasn't able to show that the inner term went to 0 as $\epsilon \to 0$ since it appears ΔK_δ gets very large for small δ. However, I believe this is probably the right way of approaching the question.

For any $\psi \in C_0^\infty(\Omega)$ we can write

$$\Delta(\psi F_{\epsilon,\delta}) = \psi \Delta F_{\epsilon,\delta} + 2\nabla\psi \cdot \nabla F_{\epsilon,\delta} + F_{\epsilon,\delta}\Delta\psi\,.$$

Supposing one could show what was discussed above, the first term will go to 0 as $\epsilon \to 0$. For the third term we have

$$\lim_{\epsilon\to 0} \int_\Omega u F_{\epsilon,\delta} \Delta\psi\,\mathrm{d}x = \int_\Omega u (\lim_{\epsilon\to 0} F_{\epsilon,\delta}) \Delta\psi\,\mathrm{d}x = 0$$

where the first equality follows from a simple application of the dominated convergence theorem since $F_{\epsilon,\delta}$ is bounded, and the second equality since $\lim_{\epsilon\to 0} F_{\epsilon,\delta} = 0$ by definition.

For the second term with the gradients we have

$$\frac{\partial}{\partial x_j} F_{\epsilon,\delta} = \begin{cases} 0 & \text{if} \quad x < e^{-\frac{1}{\epsilon}} - \delta \\ \frac{\partial}{\partial x_j} \int_{\mathbb{R}^d} f_\epsilon(y) K_\delta(x-y)\,\mathrm{d}x & \text{if} \quad e^{-\frac{1}{\epsilon}} - \delta < x < e^{-\frac{1}{\epsilon}} + \delta \\ -\frac{\epsilon x_j}{\|x\|^2} & \text{if} \quad e^{-\frac{1}{\epsilon}} + \delta < x < 1 \end{cases}.$$

Taking the limit as $\epsilon \to 0$ gives 0 for the bottom term, but once again the mollification is tricky! Unfortunately, I have no answers here.

As a final note, since $1 - F_{\epsilon,\delta} = 0$ close to the origin, we have that $\psi(1 - F_{\epsilon,\delta}) \in C_0^\infty(\Omega_0)$ for Property 1) mentioned above. As per our initial discussion, putting all of this together gives the final result that $\langle u, \Delta\psi \rangle \to 0$, thus showing u is weakly harmonic on the whole disc.

Part b)

Recall the Dirichlet problem for our domain: given the bounded open set $\Omega_0 \subset \mathbb{R}^2$ and a continuous f on the boundary $\partial\Omega_0$, find a solution to

$$\Delta u = 0 \ \text{ for } x \in \Omega_0\,, \qquad u = f \ \text{ for } x \in \partial\Omega_0\,. \tag{1.1}$$

Since Ω_0 is the punctured disc, we have

$$\partial\Omega_0 = \partial\mathcal{B}(0,1) \cup \{0\}\,.$$

Suppose we defined a continuous function

$$f(x) = \begin{cases} 0 & \text{if} \ x \in \partial\mathcal{B}(0,1) \\ 1 & \text{if} \ x = 0 \end{cases}.$$

We note here that a function on a disconnected domain, $\bigcup_\alpha U^\alpha$ of disjoint open sets U^α is continuous if it is continuous on each U^α individually, suggesting the sorts of domains for which the Dirichlet problem may fail to have a solution. By Lecture 4 we know that the solution to (1.1) on the disc Ω is given by

$$u(r,\theta) = (f * P_r)(\theta)\,,$$

where P_r is the Poisson kernel, for any $0 \leq r < 1$. Thus $u(0,\theta) = 0$ since $f = 0$ on $\partial\mathcal{B}(0,1)$. Therefore, for the solution on the punctured disc Ω_0 we must have $\lim_{r\to 0} u = 0$. But this contradicts with the choice of $f(0) = 1$, thus there is no solution to the Dirichlet problem on the punctured disc. This counterexample shows that in general a solution need not exist if the domain and boundary values don't obey nice properties.

Q2. Estimates for unbounded domains

Let $L = \sum_{|\alpha|\leq n} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$ be a constant coefficient partial differential equation operator. Recall then that L has a characteristic polynomial given by $P(\xi) = \sum_{|\alpha|\leq n} a_\alpha (2\pi i\xi)^\alpha$ such that $\widehat{Lu}(\xi) = P(\xi)\hat{u}(\xi)$. Our goal is to show that the inequality

$$\|u\|_{\mathrm{L}^2(\Omega)} \leq C\|L(u)\|_{\mathrm{L}^2(\Omega)}$$

holds for open sets $\Omega \subseteq \mathbb{R}^d$ that are unbounded. Recall that Plancherel's theorem gives us that $\|u\|_{\mathrm{L}^2(\Omega)} = \|\hat{u}\|_{\mathrm{L}^2(\mathbb{R}^d)}$ and $\|L(u)\|_{\mathrm{L}^2(\Omega)} = \|\widehat{L(u)}\|_{\mathrm{L}^2(\mathbb{R}^d)}$. This reduces the inequality to showing

$$\|u\|_{\mathrm{L}^2(\Omega)} \leq \|P(\xi)\hat{u}(\xi)\|_{\mathrm{L}^2(\mathbb{R}^d)}\,.$$

Part a)

Let $d \geq 2$. We first suppose that our unbounded domain is given by

$$\Omega = \{x \in \mathbb{R}^d \mid x_1 \in (-M, M)\}\,.$$

Let $f(x)$ be a function on Ω. Consider the Fourier transform of f given by

$$\hat{f}(\xi) = \int_\Omega f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x$$

for $\xi \in \mathbb{R}^d$. Let us extend $\hat{f}(\xi)$ into the whole complex plane (in the first coordinate) by writing

$$\hat{f}(\xi + i\eta) = \hat{f}(\xi_1 + i\eta_1, \xi_2, \ldots, \xi_d)\,,$$

which is the Fourier transform of the function

$$f(x)e^{-2\pi i x_1(\eta_1 i)} = f(x)e^{2\pi x_1\eta_1}$$

by Proposition 7.1.

Let x', ξ', Ω' denote the respective objects for the components $2, \ldots, d$. Then by Plancherel's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat{f}(\xi + i\eta)|^2 \, \mathrm{d}\xi &= \int_\Omega \left| f(x) e^{2\pi x_1 \eta_1} \right|^2 \mathrm{d}x \\ &= \int_{\Omega'} \int_{-M}^{M} |f(x)|^2 e^{4\pi x_1 \eta_1} \, \mathrm{d}x_1 \, \mathrm{d}x' \\ &\leq \int_{\Omega'} \int_{-M}^{M} |f(x)|^2 e^{4\pi M |\eta_1|} \, \mathrm{d}x_1 \, \mathrm{d}x' \\ &= e^{4\pi M |\eta_1|} \int_{\Omega'} \int_{-M}^{M} |f(x)|^2 \, \mathrm{d}x_1 \, \mathrm{d}x' \\ &= C \|f\|_{\mathrm{L}^2(\Omega)} ,\end{aligned} \tag{2.1}$$

where the third inequality follows from $e^{4\pi x_1 \eta_1} \leq e^{4\pi M |\eta_1|}$ for all $x \in (-M, M)$, and taking $C = e^{4\pi M |\eta_1|}$. If we let $f(x) = (Lu)(x)$ (which is well defined on Ω) with $\hat{f}(\xi) = P(\xi)\hat{u}(\xi)$, our new goal becomes showing that the left hand side of (2.1) is an upper bound on $\|\hat{u}(x)\|_{\mathrm{L}^2(\mathbb{R}^d)}$.

Given the characteristic polynomial $P(\xi)$ of degree n we can write $P = P_n + P'$ where P_n is a homogeneous polynomial of degree n and P' has degree less than n. The idea then is to find an orthogonal transformation O such that we can write a new coordinate axis $\zeta = O(\xi)$ where $P_n \propto \zeta^n$. I thought about this quite a lot, but can't quite say for certain that such an orthogonal transformation always exists.

- The case where $P_n(\xi) \propto \xi^n$ is trivial - we can always do this.
- My instinct is that the case where $P_n = \sum_{i=1}^d b_i \xi_i^n$ for some coefficients $b_i \in \mathbb{C}$, we can set $\zeta_1 = \sum_{i=1}^d b_i \xi_i$ to give $P_n(\zeta_1) = b_i' \zeta_1^n$ for some coefficients $b_i' \in \mathbb{C}$. For example we have $P_n = \partial_{\xi_1}^2 + \partial_{\xi_2}^2 = \partial_{\zeta_1}^2$ when $\zeta_1 = \xi_1 + \xi_2$. We can certainly always pick an orthogonal basis by applying Gram-Schmidt to the initial choice $\zeta_1 = \sum_{i=1}^d b_i \xi_i$, however I am not certain that this always reduces P_n to be in terms of ζ_1 only, and exactly how this generalises to $n > 2$ with repeated applications of the Jacobian.
- The case where there are mixed terms, e.g. $\xi_1 \xi_2 \xi_3^2$ in $P_n(\xi)$, I am quite unconvinced that an orthogonal transformation always exists to get the desired form. I am either missing something very obvious, or there is some deep algebraic geometry going on here!

Let us now proceed supposing that we have found such an O, meaning we can write $\hat{f}(\xi) = P(\xi)\hat{u}(\xi) = P(\zeta_1, \zeta')\hat{u}(\zeta_1, \zeta')$ with no mixed ζ terms. Note here that the integral in (2.1) is left unchanged under this orthogonal transformation since $|\det O| = 1$, meaning the Lebesgue measure is invariant under orthogonal transformations. In particular, this means we can denote a holomorphic function $F(z) = \hat{u}(\xi + z)$ and a polynomial $p(z) = P(\zeta_1 + z, \zeta')$ and apply Lemma 13.4. This gives

$$\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta}) F(e^{i\theta})|^2 \, \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} |P(\zeta_1 + e^{i\theta}) F(\zeta_1 + e^{i\theta})|^2 \, \mathrm{d}\theta \geq |F(0)|^2 = |\hat{u}(\xi)|^2 \tag{2.2}$$

In the left hand side of (2.1) we can use the translation invariance of the integral over $\mathbb{R}^d$ to write $\xi \mapsto \xi + \cos\theta$. Applying (2.2) to (2.1) and putting all of this together gives the desired result

$$C\|Lu\|_{\mathrm{L}^2(\Omega)} = \int_{\mathbb{R}^d} |\hat{f}(\xi + i\eta)|^2 \,\mathrm{d}\xi \geq \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi = \|\hat{u}(x)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|u\|_{\mathrm{L}^2(\Omega)} \,.$$

Thus we see that if, after a potential change in coordinates via an orthogonal transformation, at least one component has a bounded domain, the bound in (2.1) will still hold. Thus the rest of the argument will hold if this condition is met. □

Part b)

We now want to show that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq C\|L(u)\|_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $u \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ if and only if $|P(\xi)| > c > 0$ for all $\xi \in \mathbb{R}^d$. One direction is simple: suppose $|P(\xi)| > c > 0$, then by applying Plancherel's theorem a few times

$$\int_{\mathbb{R}^d} |(Lu)(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} |P(\xi)\hat{u}(\xi)|^2 \,\mathrm{d}\xi \geq c^2 \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi = c^2\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} \,.$$

For the opposite direction... I don't have much. The proof will start as supposing a contradiction, which is to say that $P(\xi)$ has a root over $\mathbb{R}^d$. My idea was to find a specific $\hat{u}$ for which it fails given we can factorise $P(\xi) = (\xi - \xi_0)Q(\xi)$ for $Q(\xi)$ degree $n-1$. The problem is, the easiest $\hat{u}$ to find is something like $\mathbb{1}(\xi \in [-1,1]^d)$, but this results in a non-compactly supported function u when taking the inverse Fourier transform.

It is truly curious to me that P having a root results in this estimate. I'll be interested to see why!