Partial Differential Equations Assignment 1

Liam Carroll - 830916

Due Date: 23rd August 2021

Q1. Niceness of heat equation

Let $f \in \mathcal{S}(\mathbb{R})$, the Schwartz space, and recall the time dependent heat equation in one dimension is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \quad \text{such that} \quad u(x,0) = f(x)\,.$$

We have shown in class that the solution to the heat equation can be expressed for $t > 0$ as

$$u(x,t) = (f * \mathcal{H}_t)(x)\,, \quad \text{where} \quad \mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} \tag{1.1}$$

is the heat kernel on the line satisfying $\mathcal{H}_t = K_{4\pi t}$, where K is the Gaussian kernel which we have shown is a good kernel. In particular, this means from Theorem 3.2 that since f is continuous for all x we have uniform convergence

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} (f * \mathcal{H}_t)(x) = f(x) = u(x,0)\,, \tag{1.2}$$

thus showing for a fixed x that $u(x,t)$ is continuous at $t = 0$. We will use this to show that continuity holds in the $\mathbb{R}^2$ sense, where we use the $d_1((x,t),(x',t')) = |x-x'|+|t-t'|$ metric.

Since $f(x)$ is uniformly continuous (since it is a Schwartz function, hence is continuous and decays at infinity), for any $\varepsilon/2 > 0$ there exists a $\delta_f > 0$ such that for any $x, x' \in \mathbb{R}$, if $|x - x'| < \delta_f$ then $|f(x) - f(x')| < \varepsilon/2$. Similarly, the uniform continuity at $t = 0$ above says that for any $\varepsilon/2$ there is a δ_0 such that for any $t \geq 0$, if $|t - 0| = |t| < \delta_0$ then $|u(x,t) - u(x,0)| = |u(x,t) - f(x)| < \varepsilon/2$. Thus, given an $\epsilon > 0$, setting $\delta = \min(\delta_f, \delta_0)$, if $\|(x,t) - (x',0)\| = |x - x'| + |t| < \delta$ then we have

$$|u(x,t) - u(x',0)| \leq |u(x,t) - f(x)| + |f(x) - f(x')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,, \tag{1.3}$$

thus showing $u(x,t)$ is uniformly continuous at $t = 0$ in the topology on $\mathbb{R}^2$ induced by d_1, which is equivalent to any other that we may desire.

We now want to show that $u(x,t)$ vanishes at infinity in the sense that $u(x,t) \to 0$ as $|x| + t \to \infty$, which we will divide into two cases - when $|x| \leq t$ and when $|x| > t$ - and by finding relevant bounds. Firstly, since $e^{-ay^2} \leq 1$ for any $y \in \mathbb{R}$ and any constant $a \geq 0$, we have

$$\begin{aligned} |u(x,t)| &= \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} f(x-y) e^{-\frac{y^2}{4t}} dy \right| \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |f(x-y)||e^{-\frac{y^2}{4t}}| dy \\ &\leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |f(x-y)| dy = \frac{C}{\sqrt{t}}\,, \end{aligned} \tag{1.4}$$

for some constant C. The final equality follows from the absolute integrability of Schwartz functions, which is easy to see since by definition there is some constant K such that $(1+x^2)|f(x)| \le K$, thus $\int_{-\infty}^{\infty} |f(x)|dx \le \int_{-\infty}^{\infty} \frac{K}{1+x^2}dx = \pi K$ (and then merely apply a change of variables on $f(y-x)$).

For the second bound, we first note that when $|y| \le |x|/2$, by the same argument as above and use of the reverse triangle inequality, we have

$$\begin{aligned}|f(x-y)| &\le \frac{K}{(1+|y-x|)^2} \le \frac{K}{(1+\big||x|-|y|\big|)^2} \le \frac{K}{(1+\frac{|x|}{2})^2} \\ &\le \frac{K}{(\frac{1}{4}(1+|x|))^2} = \frac{16K}{1+2|x|+|x|^2} \le \frac{K'}{1+|x|^2}\,, \end{aligned} \tag{1.5}$$

for some constant $K' > 0$. Thus we may write

$$\begin{aligned}|u(x,t)| &= \frac{1}{\sqrt{4\pi t}}\left|\int_{|y|\le\frac{|x|}{2}} f(x-y)e^{-\frac{y^2}{4t}}dy + \int_{|y|\ge\frac{|x|}{2}} f(x-y)e^{-\frac{y^2}{4t}}dy\right| \\ &\le \frac{1}{\sqrt{4\pi t}}\left(\int_{|y|\le\frac{|x|}{2}} |f(x-y)||e^{-\frac{y^2}{4t}}|dy + \int_{|y|\ge\frac{|x|}{2}} |f(x-y)||e^{-\frac{y^2}{4t}}|dy\right) \\ &\le \int_{|y|\le\frac{|x|}{2}} \frac{K'}{1+|x|^2}\mathcal{H}_t(y)dy + \frac{1}{\sqrt{4\pi t}}\int_{|y|\ge\frac{|x|}{2}} |f(x-y)|e^{-\frac{x^2}{16t}}dy \\ &\le \frac{K'}{1+|x|^2}\int_{-\infty}^{\infty} \mathcal{H}_t(y)dy + \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{16t}}\int_{-\infty}^{\infty} |f(x-y)|dy \\ &\le \frac{C}{1+|x|^2} + \frac{C}{\sqrt{t}}e^{-\frac{x^2}{16t}}\,, \end{aligned} \tag{1.6}$$

for some constant $C > 0$. Note that the second line follows from the triangle inequality, the third by our bound in (1.5), the fourth by expanding the domain of integration of positive functions, and the last by the absolute integrability of the heat kernel and Schwartz functions.

Therefore, as $|x|+t \to \infty$, in the subset of $\mathbb{R}^2$ where $|x| \le t$ we have $|u(x,t)| \le \frac{C}{\sqrt{t}} \to 0$ as $t \to \infty$, and in the subset where $|x| \ge t$, for any fixed t we have $|u(x,t)| \le \frac{C}{1+|x|^2} + \frac{C}{\sqrt{t}}e^{-\frac{x^2}{16t}} \to 0$ as $|x| \to \infty$. Thus $u(x,t)$ decays at infinity as desired. $\square$

Q3. Uniqueness of harmonic functions on the strip

Let $u(x, y)$ be a solution to Laplace's equation, $\Delta u = 0$, in the infinite strip Z where

$$Z = \{(x, y) : 0 < y < 1, \ \ -\infty < x < \infty\} .$$

Further, suppose $u(x, 0) = u(x, 1) = 0$, $u(x, y)$ is continuous on the closure $\overline{Z}$ and $u(x, y)$ vanishes as $|x| \to \infty$. We will show that this necessarily implies that $u = 0$ in Z.

For a contradiction, suppose there is a point $(x_0, y_0) \in Z$ such that $|u(x_0, y_0)| > 0$. Then since u vanishes at infinity we may find a radius R, defining a closed disc restricted to Z

$$D_R = \{(x, y) : x^2 + y^2 \leq R^2 \, , \ \ 0 \leq y \leq 1\} ,$$

such that for all (x, y) outside of D_R we have $|u(x, y)| \leq \frac{|u(x_0,y_0)|}{2}$. Then since u is continuous on the compact set $\overline{Z}$ it necessarily attains a maximum on D_R, say at the point $(x_1, y_1) \in D_R$ with $|u(x_1, y_1)| = M$ for some $M > 0$. Then $|u(x, y)| \leq M$ for all $(x, y) \in D_R$, and in particular $|u(x_0, y_0)| \leq M$. By our assumption on D_R this further implies that $|u(x, y)| \leq M/2$ for all points outside of D_R, thus showing $|u(x, y)|$ is bounded by M on all of Z.

Recall the mean value property of harmonic functions from Proposition 6.4, which tells us that if a closed disc D_r of radius $r > 0$ centred at (x_1, y_1) is contained in Z, then

$$u(x_1, y_1) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r\cos\theta, y_1 + r\sin\theta) d\theta .$$

In order for D_r to be contained in Z we require $0 \leq r < \min(|y_1|, |1 - y_1|)$ since the y domain of Z is $y \in (0, 1)$. We can then show by contradiction that the integrand is constant on all of the circle. Let r be fixed satisfying the above constraint, and suppose there was some arc of length $\delta > 0$, parametrised by an interval $\theta \in (\theta_0, \theta_1)$, such that $u(x_1 + r\cos\theta, y_1 + r\sin\theta) \leq M - \varepsilon$ for some $\varepsilon > 0$. Then using the ML bound we have

$$\begin{aligned} |u(x_1, y_1)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} u(x_1 + r\cos\theta, y_1 + r\sin\theta) d\theta \right| \\ &\leq \frac{1}{2\pi} \left(\int_{\theta_0}^{\theta_1} |u(x_1 + r\cos\theta, y_1 + r\sin\theta)| d\theta + \int_{S^1\setminus(\theta_0,\theta_1)} |u(x_1 + r\cos\theta, y_1 + r\sin\theta)| d\theta \right) \\ &\leq \frac{1}{2\pi} \big((M - \varepsilon)\delta + M(2\pi - \delta) \big) = M - \frac{\varepsilon\delta}{2\pi} < M , \end{aligned}$$

but by assumption $u(x_1, y_1) = M$, hence arriving at a contradiction. Thus for all $\theta \in [0, 2\pi)$ we have $u(x_1 + r\cos\theta, y_1 + r\sin\theta) = M$. We may then take the limit of r to the boundary, thus $r \to \min(|y_1|, |1-y_1|)$, and using the continuity of u on $\overline{Z}$ we see that $u(x_1, 0) = M$ if $0 \leq y_1 \leq \frac{1}{2}$ or $u(x_1, 1) = M$ if $\frac{1}{2} < y_1 \leq 1$. But by definition we have $u(x, 0) = u(x, 1) = 0$ for all $x \in \mathbb{R}$, thus showing $M = 0$, which shows $u(x, y) = 0$ since M bounds u everywhere. □

We remark that this result demonstrates uniqueness of solutions of Laplace's equation with prescribed boundary conditions due to the linearity of Δ. That is, if u_1 and u_2 both satisfy $\Delta u_1 = \Delta u_2 = 0$, $u_1(x, 0) = u_2(x, 0) = a$ and $u_1(x, 1) = u_2(x, 1) = b$ then we may set $v = u_1 - u_2$ with $\Delta v = 0$, $v(x, 0) = 0$ and $v(x, 1) = 0$, thus applying our above result shows that $v = 0$, thus $u_1 = u_2$.