Lie Algebras Assignment 4

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Lecture 6

Q4. Commutator and the exponential

Let $X, Y \in \mathfrak{gl}(n, \mathbb{C})$. We will prove the following two identities:

$$[Y, X] = \frac{\partial^2}{\partial s \partial t} \left(\exp(-sY) \exp(-tX) \exp(sY) \exp(tX) \right) \Big|_{s=t=0} \,, \tag{4.1}$$

$$\text{and} \quad \exp(-tY) \exp(-tX) \exp(tY) \exp(tX) = \exp(t^2 [Y, X] + O(t^3)) \,, \tag{4.2}$$

where $[Y, X] = YX - XY$ is the commutator.

Part a)

We calculate

$$\begin{aligned} \frac{\partial}{\partial t} \left(e^{-sY} e^{-tX} e^{sY} e^{tX} \right) &= \left(\frac{\partial}{\partial t} \left(e^{-sY} e^{-tX} \right) \right) \left(e^{sY} e^{tX} \right) + \left(e^{-sY} e^{-tX} \right) \left(\frac{\partial}{\partial t} \left(e^{sY} e^{tX} \right) \right) \\ &= -e^{-sY} X e^{-tX} e^{sY} e^{tX} + e^{-sY} e^{-tX} e^{sY} X e^{tX} \,, \end{aligned}$$

so taking $\frac{\partial}{\partial s}$ of the above expression gives

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} \left(e^{-sY} e^{-tX} e^{sY} e^{tX} \right) &= \left(\frac{\partial}{\partial s} \left(-e^{-sY} X e^{-tX} \right) \right) \left(e^{sY} e^{tX} \right) + \left(-e^{-sY} X e^{-tX} \right) \left(\frac{\partial}{\partial s} \left(e^{sY} e^{tX} \right) \right) \\ &\quad + \left(\frac{\partial}{\partial s} \left(e^{-sY} e^{-tX} \right) \right) \left(e^{sY} X e^{tX} \right) + \left(e^{-sY} e^{-tX} \right) \left(\frac{\partial}{\partial s} \left(e^{sY} X e^{tX} \right) \right) \\ &= e^{-sY} Y X e^{-tX} e^{sY} e^{tX} - e^{-sY} e^{-tX} X Y e^{sY} e^{tX} \\ &\quad - Y e^{-sY} e^{-tX} e^{sY} X e^{tX} + e^{-sY} e^{-tX} e^{sY} Y X e^{tX} \,, \end{aligned}$$

where we have used $\frac{\partial}{\partial t} e^{tX} = X e^{tX} = e^{tX} X$ many times over. Therefore using the fact that $e^{0X} = 1$, the identity operator, we have

$$\frac{\partial^2}{\partial s \partial t} \left(e^{-sY} e^{-tX} e^{sY} e^{tX} \right) \Big|_{s=t=0} = YX - XY - YX + YX = YX - XY = [Y, X] \,. \tag{4.3}$$

Part b)

We remark that this statement will only be valid for $\|X\|, \|Y\| < \log 2$ to ensure that we can apply the logarithm at the end, so suppose X and Y satisfy this hypothesis. By

Taylor's theorem (with Lagrange remainder, where $\frac{d^k}{dx^k}e^x = e^x$ for any $k \in \mathbb{N}$) we have that for $t \in \mathbb{R}$ and some bounded operator X that there exists some $b \in [0,t]$ such that

$$\exp(t) = 1 + t + \frac{1}{2}t^2 + \frac{\exp(b)}{6}t^3\,,$$

$$\text{so} \quad \left|\exp(\|X\|t) - 1 - \|X\|t - \frac{1}{2}\|X\|^2t^2\right| = \frac{\exp(b)}{6}\|X\|^3t^3\,, \tag{4.4}$$

so $\sum_{j=3}^{\infty} \frac{(\|X\|t)^j}{j!} = \frac{\exp(b)}{6}\|X\|^3t^3$. But then we have

$$\left\|\exp(Xt) - 1 - Xt - \frac{1}{2}X^2t^2\right\| = \left\|\sum_{j=3}^{\infty}\frac{(Xt)^j}{j!}\right\| \le \sum_{j=3}^{\infty}\frac{(\|X\|t)^j}{j!} = \frac{\exp(b)\|X\|^3}{6}t^3\,. \tag{4.5}$$

We recall from the Big O notation remark that $f(t) = O(t^3)$ means that there exists some $C > 0$ and $\varepsilon > 0$ such that whenever $t < \varepsilon$ we have $\|f(t)\| \le Ct^3$. So, since $\|X\| < \log 2$ by hypothesis we have for sufficiently small t that $\exp(b) \le \exp(\|X\|t) \le \exp(\|X\|) \le \exp(\log 2) = 2$, so

$$\left\|\exp(Xt) - 1 - Xt - \frac{1}{2}X^2t^2\right\| \le \frac{\exp(b)\|X\|^3}{6}t^3 \le \frac{2(\log 2)^3}{6}t^3 = \frac{(\log 2)^3}{3}t^3\,,$$

which we can write as

$$\exp(Xt) = 1 + Xt + \frac{1}{2}X^2t^2 + O(t^3) \tag{4.6}$$

by our definition of $O(t^3)$.

Let us denote

$$R_3(X) = \exp(Xt) - 1 - Xt - \frac{1}{2}X^2t^2 = \sum_{j=3}^{\infty}\frac{(Xt)^j}{j!}\,. \tag{4.7}$$

Then we know from our above analysis that $\|R_3(X)\| \le (\log 2)^3t^3$ for any $\|X\| \le \log 2$. Suppose X and Y satisfy this condition and define $\mathcal{L} = R_3(Y)$ and $\mathcal{R} = R_3(X)$. Then we calculate

$$\begin{aligned}
\exp(tY)\exp(tX) &= (1 + Yt + \frac{1}{2}Y^2t^2 + \mathcal{L})(1 + Xt + \frac{1}{2}X^2t^2 + \mathcal{R}) \\
&= 1 + Xt + \frac{1}{2}X^2t^2 + \mathcal{R} + Yt + YXt^2 + \frac{1}{2}YX^2t^3 + Y\mathcal{R}t \\
&\quad + \frac{1}{2}Y^2t^2 + \frac{1}{2}Y^2Xt^3 + \frac{1}{4}Y^2X^2t^4 + \frac{1}{2}Y^2\mathcal{R}t^2 + \mathcal{L} + \mathcal{L}Xt + \frac{1}{2}\mathcal{L}X^2t^2 + \mathcal{L}\mathcal{R}\,,
\end{aligned}$$

meaning we can calculate

$$\begin{aligned}
&\|e^{tY}e^{tX} - 1 - (X+Y)t - (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2\| \\
&= \|\mathcal{R} + \frac{1}{2}YX^2t^3 + Y\mathcal{R}t + \frac{1}{2}Y^2Xt^3 + \frac{1}{4}Y^2X^2t^4 + \frac{1}{2}Y^2\mathcal{R}t^2 + \mathcal{L} + \mathcal{L}Xt + \frac{1}{2}\mathcal{L}X^2t^2 + \mathcal{L}\mathcal{R}\| \\
&\le \|\mathcal{R}\| + \frac{1}{2}\|Y\|\|X\|^2t^3 + \|Y\|\|\mathcal{R}\|t + \frac{1}{2}\|Y\|^2\|X\|t^3 + \frac{1}{4}\|Y\|^2\|X\|^2t^4 + \frac{1}{2}\|Y\|^2\|\mathcal{R}\|t^2 \\
&\quad + \|\mathcal{L}\| + \|\mathcal{L}\|\|X\|t + \frac{1}{2}\|\mathcal{L}\|\|X\|^2t^2 + \|\mathcal{L}\|\|\mathcal{R}\| \\
&\le (\log 2)^3t^3 + \frac{1}{2}(\log 2)^3t^3 + (\log 2)^4t^4 + \frac{1}{2}(\log 2)^3t^3 + \frac{1}{4}(\log 2)^4t^4 + \frac{1}{2}(\log 2)^5t^5 \\
&\quad + (\log 2)^3t^3 + (\log 2)^4t^4 + \frac{1}{2}(\log 2)^5t^5 + (\log 2)^6t^6 \\
&\le 10(\log 2)^3t^3\,,
\end{aligned} \tag{4.8}$$

where the last inequality holds for sufficiently small t such that $t^3 < t^4 < t^5 < t^6$. Thus we may write

$$e^{tY}e^{tX} = 1 + (X+Y)t + (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2 + O(t^3)\,, \tag{4.9}$$
$$\text{and} \quad e^{-tY}e^{-tX} = 1 - (X+Y)t + (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2 + O(t^3)\,.$$

We can then perform a crude calculation that is justified using an identical kind of analysis as in (4.8), with all the same hypotheses and bounds, to see that

$$\begin{aligned}
e^{-tY}e^{-tX}e^{tY}e^{tX} &= \left(1 - (X+Y)t + (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2 + O(t^3)\right)\left(1 + (X+Y)t\right.\\
&\quad \left. + (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2 + O(t^3)\right)\\
&= 1 + (X+Y)t + (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2 - (X+Y)t - (X+Y)^2t^2\\
&\quad + (\frac{1}{2}X^2 + YX + \frac{1}{2}Y^2)t^2 + O(t^3)\\
&= 1 + \left(X^2 + 2YX + Y^2 - X^2 - XY - YX - Y^2\right)t^2 + O(t^3)\\
&= 1 + [Y,X]t^2 + O(t^3)\,.
\end{aligned} \tag{4.10}$$

Then we see that $\lim_{t\to 0} e^{-tY}e^{-tX}e^{tY}e^{tX} = 1$ by the continuity of the exponential, which tells us that for t sufficiently small we have $\|e^{-tY}e^{-tX}e^{tY}e^{tX} - 1\| = \|[Y,X]t^2 + O(t^3)\| < 1$, allowing us to take the logarithm of both sides due to the hypothesis that $\|X\|, \|Y\| < \log 2$, thus meaning our expression fits inside the domain. Therefore,

$$\begin{aligned}
\log(e^{-tY}e^{-tX}e^{tY}e^{tX}) &= \log(1 + [Y,X]t^2 + O(t^3)) = [Y,X]t^2 + O(\|[Y,X]t^2 + O(t^3)\|^2)\\
&= [Y,X]t^2 + O(t^4)\,.
\end{aligned}$$

For sufficiently small t we have $\|\log(e^{-tY}e^{-tX}e^{tY}e^{tX}) - [Y,X]t^2\| \leq Ct^4 \leq Ct^3$ for some constant C, so we may replace the $O(t^4)$ with $O(t^3)$ in line with the question. Hence taking the exponential of both sides (which is valid by Lemma B1-14) we have

$$e^{-tY}e^{-tX}e^{tY}e^{tX} = \exp([Y,X]t^2 + O(t^3))\,. \tag{4.11}$$

□

Q5. Fullness of Lie functor

Let G be a matrix Lie group where every element $g \in G$ can be written as

$$g = \exp(X_1) \dots \exp(X_n) \quad \text{for some} \quad X_1, \dots, X_n \in \mathfrak{g}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of G - that is to say, G is connected. Recall the functor defined in lectures

$$T : \mathrm{rep}(G) \longrightarrow \mathrm{rep}(\mathfrak{g}) \,, \tag{5.1}$$
$$X.v = \frac{d}{dt}\left(\exp(tX).v\right)\Big|_{t=0} = \lim_{t\to 0} \frac{\exp(tX).v - v}{t} \,,$$

which sends a representation $\exp(tX).v$ of G to a representation of $\mathfrak{g}$ given by $X.v$ above. We want to show that T is full, that is it is surjective on morphisms. In other words, if $(V, ._V)$ and $(W, ._W)$ are representations of G and $\phi : V \to W$ is a linear morphism of $\mathfrak{g}$-representations, then it is also a morphism of G-representations. Note that while the $._V$ and $._W$ notation is cumbersome, we adopt it in this proof to ensure utmost clarity when dealing with many different operations.

We begin by getting all of our notation in order. Since ϕ is a linear morphism of $\mathfrak{g}$-representations we know that

$$\phi(X._V v) = X._W \phi(v) \quad \text{for all} \quad X \in \mathfrak{g} \,, \quad v \in V \,. \tag{5.2}$$

We want to show that for any $g \in G$ we have $\phi(g._V v) = g._W \phi(v)$ where ϕ is the same $\mathfrak{g}$-representation now acting on elements of G. We start with the base case where we let $g = \exp(X) \in G$ for some $X \in \mathfrak{g}$, so we want to show $\phi(\exp(X)._V v) = \exp(X)._W \phi(v)$. Recall that for any representation $._V$, for any $g \in G$ our action $g._V v$ can also be denoted by an endomorphism $\alpha_g \in \mathrm{End}(V, V)$ where $\alpha_g(v) = g._V v$ To this end we can define the following functions

$$f : \mathbb{R} \longrightarrow \mathrm{End}(V, W) \,, \quad f(t) = \phi \circ \alpha_{\exp(tX)} \,, \tag{5.3}$$
$$g : \mathbb{R} \longrightarrow \mathrm{End}(V, W) \,, \quad g(t) = \alpha_{\exp(tX)} \circ \phi \,,$$

where $\circ$ is composition of endomorphisms (i.e. matrix multiplication), and clearly in the second case we let $\alpha_{\exp(tX)} \in \mathrm{End}(W, W)$. Hence for any $v \in V$ we have

$$f(t)(v) = (\phi \circ \alpha_{\exp(tX)})(v) = \phi(\alpha_{\exp(tX)}(v)) = \phi(\exp(tX)._V v) \,, \tag{5.4}$$
$$g(t)(v) = (\alpha_{\exp(tX)} \circ \phi)(v) = \alpha_{\exp(tX)}(\phi(v)) = \exp(tX)._W \phi(v) \,.$$

We have now reduced our base case to showing that $f = g$, which we can do by showing they satisfy the same differential equation.

Recall that if X and Y commute then $\exp(X + Y) = \exp(X)\exp(Y) = \exp(Y)\exp(X)$, and so since tX and hX obviously commute for scalars t and h, we have

$$\exp((h+t)X)._V v = \exp(hX + tX)._V v = (\exp(hX)\exp(tX))._V v = \exp(hX)._V(\exp(tX)._V v) \,,$$

where we used property R1 in the definition of a G-representation. We can then calculate

$$\begin{aligned}
\frac{d}{dt} f(t)(v) = \frac{d}{dt}\Big((\phi \circ \alpha_{\exp(tX)})(v)\Big) &= \lim_{h\to 0} \frac{\phi(\exp((t+h)X)._V v) - \phi(\exp(tX)._V v)}{h} \\
&= \phi\left(\lim_{h\to 0} \frac{\exp(hX)._V(\exp(tX)._V v) - \exp(tX)._V v}{h}\right) \\
&= \phi(X._V(\exp(tX)._V v)) \\
&= X._W \phi(\exp(tX)._V v) = X._W((\phi \circ \alpha_{\exp(tX)})(v)) \,.
\end{aligned}$$

In the second equality we used the linearity (and hence continuity) of ϕ, in the third equality we used the definition of the $\mathfrak{g}$-representation from (5.1), and in the fourth equality we used the fact that ϕ is a morphism of G representations. Therefore f satisfies the differential equation

$$\frac{d}{dt} f(t) = X._W f(t)\,. \tag{5.5}$$

Similarly we can calculate

$$\begin{aligned}
\frac{d}{dt} g(t)(v) = \frac{d}{dt}\Big((\alpha_{\exp(tX)} \circ \phi)(v)\Big) &= \frac{d}{dt}\big(\exp(tX)._W \phi(v)\big) \\
&= \lim_{h\to 0} \frac{\exp((t+h)X)._W \phi(v) - \phi(v)}{h} \\
&= \lim_{hto0} \frac{\exp(hX)._W(\exp(tX)._W \phi(v)) - \phi(v)}{h} \\
&= X._W(\exp(tX)._W \phi(v)) = X._W(\alpha_{\exp(tX)} \circ \phi)(v)\,,
\end{aligned}$$

so once again we have

$$\frac{d}{dt} g(t) = X._W g(t)\,. \tag{5.6}$$

Finally, notice that $f(0) = \phi \circ \alpha_{\exp(0)} = \phi$ and $g(0) = \alpha_{\exp(0)} \circ \phi = \phi$, so we have shown that f and g satisfy the differential equation

$$\begin{cases} \frac{d}{dt} y(t) = X._W y(t) \\ y(0) = \phi \end{cases}, \tag{5.7}$$

and so by Picard's theorem, using the same justification as in Theorem L4-5, we know that the solution $y(t)$ is *unique*, hence $f(t) = g(t)$. Evaluating at $t = 1$ we have

$$f(1)(v) = \phi(\exp(X)._V v) = \exp(X)._W \phi(v) = g(1)(v)\,, \tag{5.8}$$

which concludes the base case. The inductive step is easy though: suppose this holds for $X_1, \ldots, X_n \in \mathfrak{g}$ so

$$\phi((\exp(X_1)\ldots\exp(X_n))._V v) = (\exp(X_1)\ldots\exp(X_n))._W \phi(v)\,. \tag{5.9}$$

Recall that for a G-representation we have for all $g, h \in G$ and $v \in V$ that $g.(h.v) = (gh).v$, so for $X_{n+1} \in \mathfrak{g}$ we have

$$\begin{aligned}
\phi\big((\exp(X_1 \ldots \exp(X_n)\exp(X_{n+1}))._V v\big) &= \phi\big((\exp(X_1)\ldots\exp(X_n))._V(\exp(X_{n+1})._V v)\big) \\
&= (\exp(X_1)\ldots\exp(X_n))._W \phi(\exp(X_{n+1})._V v) \\
&= (\exp(X_1)\ldots\exp(X_n))._W(\exp(X_{n+1})._W \phi(v)) \\
&= (\exp(X_1)\ldots\exp(X_n)\exp(X_{n+1}))._W \phi(v)\,,
\end{aligned}$$

where we used the inductive hypothesis in the second equality and (5.8) in the third. Thus we have shown that for any $\exp(X_1)\ldots\exp(X_n) = g \in G$ we have

$$\phi(g._V v) = g._W \phi(v) \tag{5.10}$$

and so ϕ is also a morphism of G-representations, thus T is full. $\square$