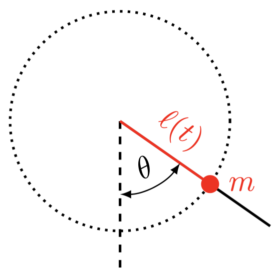
Advanced Methods: Transforms Assignment 1

Liam Carroll - 830916

Due date: 1st May 2020

Q1. Parametric resonance

Consider a point-like mass m that can move up and down along a rod that rotates. The position of the rod is described by an angle $\theta = \theta(t)$ that measures the deviation of the rod from the vertical as depicted in the assignment diagram. The position of the mass along the rod can be described by $\ell = \ell(t)$ which we regard as being a fixed, and given, function of time. Assume friction is negligible.



Part a)

The mass m has position $\boldsymbol{x} = (x, y)$ in cartesian coordinates.
Since $\ell(t)$ is a fixed function, this tells us that ℓ does *not* correspond to a degree of freedom in the system. Hence, we only have the one generalised coordinate corresponding to the angular degree of freedom, which we can denote as $\{q_1\} = \{\theta\}$. Despite the fact that ℓ is not a generalised coordinate, its time dependence means that time derivatives of ℓ still occur in calculations. We consider the xy-plane as emanating from the suspension point in the standard way (i.e. $x > 0$ is "right" and $y > 0$ is "up"). Hence, this gives us

$$x = \ell \sin\theta\,, \qquad y = -\ell\cos\theta\,. \tag{1.1}$$

We can then calculate the respective time derivatives,

$$\dot{x} = \dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta\,, \tag{1.2}$$

$$\dot{y} = -\dot{\ell}\cos\theta + \ell\dot{\theta}\sin\theta\,, \tag{1.3}$$

where $\boldsymbol{v} = \dot{\boldsymbol{x}} = (\dot{x}, \dot{y})$ denotes the velocity of the mass m.

Part b)

We now wish to calculate the Lagrangian $L = L(t, \{q_i(t)\}, \{\dot{q}_i(t)\}) = T - V$ where we have V being conservative (since gravitational potential does not depend on generalised velocities). We can calculate the kinetic energy,

$$\begin{aligned} T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) &= \frac{1}{2}m\left((\dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta)^2 + (-\dot{\ell}\cos\theta + \ell\dot{\theta}\sin\theta)^2\right) \\ &= \frac{m}{2}\Big((\dot{\ell}^2\sin^2\theta + 2\dot{\ell}\ell\dot{\theta}\sin\theta\cos\theta + \ell^2\dot{\theta}^2\cos^2\theta) \\ &\qquad + (\ell^2\dot{\theta}^2\sin^2\theta - 2\dot{\ell}\ell\dot{\theta}\sin\theta\cos\theta + \dot{\ell}^2\cos^2\theta)\Big) \\ &= \frac{m}{2}(\dot{\ell}^2 + \ell^2\dot{\theta}^2)\,. \end{aligned} \tag{1.4}$$

Similarly, we can calculate the potential energy,

$$V = mgy = -mg\ell\cos\theta\,. \tag{1.5}$$

Thus we can then calculate the Lagrangian of the system,

$$L = T - V = m\left(\frac{1}{2}\dot{\ell}^2 + \frac{1}{2}\ell^2\dot{\theta}^2 + g\ell\cos\theta\right)\,. \tag{1.6}$$

Part c)

We now wish to simplify the equations of motion for the system using the Euler-Lagrange equations, namely

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}\,, \tag{1.7}$$

where we only have one equation due to having one generalised coordinate $q_1 = \theta$. Thus we can calculate

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}\,, \quad \text{so} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2m\dot{\ell}\ell\dot{\theta} + m\ell^2\ddot{\theta}\,, \tag{1.8}$$

and

$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta\,. \tag{1.9}$$

Hence, combining (1.8) and (1.9) into (1.7), we get

$$2m\dot{\ell}\ell\dot{\theta} + m\ell^2\ddot{\theta} = -mg\ell\sin\theta\,,$$

and so with appropriate cancellation and rearrangement, we arrive at the governing equation of motion for the system

$$\boxed{\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta} + g\sin\theta = 0}\,. \tag{1.10}$$

A numerical solution to this second-order ODE (with necessary initial conditions) could be found quite easily with standard ODE computation - we will leave this for another time. We also notice that in the case $\ell(t)$ being constant, i.e. $\dot{\ell} = 0$, this equation produces the standard model for a simple pendulum.

Part d)

Given that the system is a (relatively) simple pendulum, one may expect there to be a conserved charge. In particular, it "feels" like it may be a closed system, and since the Lagrangian does not *appear* to explicitly depend on time, one would like to think that there is a conserved charge that arises from time invariance.

However, after attempting to show that $dH/dt = 0$, it became apparent why this was not true. The fact that $\ell = \ell(t)$ is a fixed function that depends on time means that the Lagrangian is indeed time dependent - i.e. there is no time symmetry that arises from this. Namely, in calculating dH/dt we find that

$$\begin{aligned}\frac{dH}{dt} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L\right) \\ &= m\dot{\ell}\ell\dot{\theta}^2 + m\ell^2\dot{\theta}\ddot{\theta} - m\dot{\ell}\ddot{\ell} - g\dot{\ell}\cos\theta + g\ell\dot{\theta}\sin\theta\,,\end{aligned} \tag{1.11}$$

and then using the equations of motion from (1.10), this becomes

$$\frac{dH}{dt} = \dot{\ell}\left(-m\ell\dot{\theta}^2 - m\ddot{\ell} - g\cos\theta\right)\,, \tag{1.12}$$

which is clearly non-zero! However, we again note that in the case of a simple pendulum with $\dot{\ell} = 0$, this returns $dH/dt = 0$ which means energy is conserved in the simple pendulum system - this gives us a big hint that the pesky $\ell(t)$ is what is causing our lack of symmetry.

Clearly, since there are no cyclic coordinates, momentum is not conserved either. It remains to do a sanity check on the angular momentum. After a simple calculation, we see that

$$\frac{d}{dt}\boldsymbol{J} = \frac{d}{dt}(mx\dot{y} - m\dot{x}y) = \frac{d}{dt}(m\ell^2\dot{\theta}\boldsymbol{k}) = (2m\dot{\ell}\ell\dot{\theta} + m\ell^2\ddot{\theta})\boldsymbol{k}\,, \tag{1.13}$$

but by (1.10), this is

$$\frac{d}{dt}\boldsymbol{J} = -mg\ell\sin\theta\boldsymbol{k} \neq 0\,, \tag{1.14}$$

and so once again we see that angular momentum is *not* conserved.

Putting all of this together, we conclude that there are *no conserved charges* for this system, and this is largely down to the fact that $\ell(t)$ causes these lacks of symmetry. Emmy Noether is sad at this result.

Q2. Probabilistic application of Maximum Entropy Principle

Let X be a random variable with probability density $\rho : [a, b] \to [0, \infty)$ where

$$P(X \leq x) = \int_a^x \rho(x')dx', \qquad \text{with} \qquad \int_a^b \rho(x)dx = 1. \tag{2.1}$$

Part a)

We want find the distribution ρ that maximises the entropy given these constraints. The entropy of ρ in our case is defined as

$$S[\rho(x)] = -\int_a^b \rho(x) \log \rho(x)\, dx. \tag{2.2}$$

We clearly wish to maximise the value of the functional $S[\rho(x)]$ subject to the normalisation integral constraint in (2.1). We do this by freely optimising the extended functional

$$\mathcal{F}[\lambda, \rho(x)] = \int_a^b [-\rho(x) \log \rho(x) - \lambda\rho(x)]\, dx + \lambda, \tag{2.3}$$

where the Lagrange multiplier λ is a parameter and not a function. Let $f(x, \rho(x)) = -\rho(x) \log \rho(x)$ and $g(x, \rho(x)) = \rho(x)$. Then, from the lecture notes, we can use the modified Euler-Lagrange equations for k integral constraints, namely

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \rho'}\right) - \frac{\partial f}{\partial \rho} = \lambda \sum_{i=1}^{k} \left[\frac{d}{dx}\left(\frac{\partial g_i}{\partial \rho'}\right) - \frac{\partial g_i}{\partial \rho}\right]. \tag{2.4}$$

Performing these trivial calculations, particularly due to the lack of ρ' dependence, we arrive at the Euler-Lagrange equations for the system,

$$\log \rho(x) + 1 + \lambda = 0. \tag{2.5}$$

Solving for $\rho(x)$ we see that $\rho(x) = e^{-1-\lambda}$, but since λ is a constant parameter, this implies that $\rho(x)$ is constant. Since the support on $\rho(x)$ is the compact interval $[a, b]$, we can write

$$\rho(x) = C\mathbb{1}(a \leq x \leq b),$$

for a normalising constant C. With great ease we see that

$$1 = C \int_{-\infty}^{\infty} \mathbb{1}(a \leq x \leq b)dx = C \int_a^b dx = C(a - b) \quad \text{so} \quad C = \frac{1}{a - b}.$$

Thus, the distribution ρ that maximises the entropy is the uniform distribution on $[a, b]$,

$$\boxed{\rho(x) = \frac{1}{b - a}\mathbb{1}(a \leq x \leq b)}. \tag{2.6}$$

□

Part b)

Consider the same scenario as above, but this time with the additional integral constraint of fixing the mean to have a value $\mu \in (a, b)$, that is,

$$\int_a^b x\rho(x)dx = \mu\,. \tag{2.7}$$

Again, we wish to maximise the entropy $S[\rho(x)]$ as in (2.2). This time we wish to optimise the functional

$$\mathcal{F}[\lambda, \rho(x)] = \int_a^b \left[-\rho(x)\log\rho(x) - \lambda_1\rho(x) - \lambda_2 x\rho(x)\right]dx + \lambda_1 + \lambda_2\mu\,, \tag{2.8}$$

using (2.4) once again for Lagrange multiplier parameters λ_1, λ_2. Performing similar calculations, this time with the additional function of $g_2(x, \rho(x)) = x\rho(x)$ and substituting into (2.4), we arrive at

$$\log\rho(x) + 1 + \lambda_1 + \lambda_2 x = 0\,. \tag{2.9}$$

Rearranging for $\rho(x)$, we get

$$\rho(x) = e^{-1-\lambda_1-\lambda_2 x} = Ae^{-\lambda x}\,, \tag{2.10}$$

for constants A and λ. Applying the normalisation constraint gives us

$$\int_a^b \rho(x)dx = A\int_a^b e^{-\lambda x}dx = \frac{A}{\lambda}\left(e^{-\lambda a} - e^{-\lambda b}\right) = 1\,, \tag{2.11}$$

and then the fixed mean constraint gives us (using integration by parts),

$$\begin{aligned}\mu = \int_a^b x\rho(x)dx = A\int_a^b xe^{-\lambda x}dx &= A\left[-\frac{1}{\lambda}\left(x + \frac{1}{\lambda}\right)e^{-\lambda x}\right]_a^b \\ &= A\left[\frac{1}{\lambda}\left(a + \frac{1}{\lambda}\right)e^{-\lambda a} - \frac{1}{\lambda}\left(b + \frac{1}{\lambda}\right)e^{-\lambda b}\right] \\ &= \frac{A}{\lambda}\left[\left(ae^{-\lambda a} - be^{-\lambda b}\right) + \frac{1}{\lambda}\left(e^{-\lambda a} - e^{\lambda b}\right)\right]\,.\end{aligned}$$

If we then use (2.11), we arrive at

$$\mu = \frac{ae^{-\lambda a} - be^{-\lambda b}}{e^{-\lambda a} - e^{-\lambda b}} + \frac{1}{\lambda}\,. \tag{2.12}$$

Clearly, this is not a closed form solution for the constant λ in terms of μ - far from it. Indeed, there is no closed form solution for this equation, which when plotted resembles logistic-curve behaviour. However, we can at least verify that there is *a* solution to (2.12) for $\mu \in (a, b)$.

Consider the function $f : (-\infty, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{ae^{-xa} - be^{-xb}}{e^{-xa} - e^{-xb}} + \frac{1}{x} = \frac{a - be^{-(b-a)x}}{1 - e^{-(b-a)x}} + \frac{1}{x}\,. \tag{2.13}$$

With elementary calculations and L'Hôpital's rule, we see that

$$\lim_{x\to-\infty} f(x) \overset{[\frac{\infty}{\infty}]}{\underset{\mathrm{H}}{=}} \frac{b(b-a)e^{-(b-a)x}}{(b-a)e^{-(b-a)x}} = b\,, \tag{2.14}$$

and similarly

$$\lim_{x\to\infty} f(x) = a\,. \tag{2.15}$$

My original intention was also to show that $f(x)$ is monotonically decreasing, which it is, however that derivation is not really worth the pain of showing so we omit it. Thus, with all of these properties combined, we see that f is a bijection from $(-\infty, \infty) \to (a, b)$ - in other words, we are *guaranteed* to have a unique solution for λ in (2.12), even if we must compute it numerically.

Putting all of this together, we see that the distribution ρ on a fixed closed interval $[a, b]$ that maximises the entropy given a fixed mean μ is

$$\boxed{\rho(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a} - e^{-\lambda b}} \mathbb{1}(a \leq x \leq b)}\,, \tag{2.16}$$

where λ is the *unique* solution to (2.12). We see that this is merely an exponential distribution with rate λ that is appropriately normalised to its truncated interval.

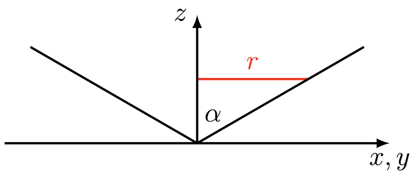
□

N.B. - It is interesting to note that $\lim_{\lambda\to 0} \rho(x) = 0$, *which may seem like a trivial point, however turns out to have an important implication. Due to the symmetry of* $f(x)$, *it is possible to show that* $lim_{x\to 0} f(x) = (a + b)/2$. *This tells us that in the case of* $\mu = (a + b)/2$, *the distribution that maximises entropy does not exist (since* $\rho(x) = 0$ *cannot be a probability distribution).*

Q3. Geodesics on a cone

The surface of an infinite cone in $\mathbb{R}^3$ is specified by the two equations

$$\begin{aligned} x^2 + y^2 &= r^2 \\ z &= r\cot\alpha \\ \text{where} \quad z &\geq 0\,. \end{aligned} \tag{3.1}$$



Part a)

Cylindrical coordinates (i.e. polar coordinates in the xy-plane) are the natural system to view the problem, so we let $\boldsymbol{x} = (x, y, z) = (r\cos\theta, r\sin\theta, z)$. We wish to express the infinitesimal line element $ds^2 = dx^2 + dy^2 + dz^2$ in terms of these new orthogonal coordinates. We know from vector calculus that for a new orthogonal coordinate system (q_1, q_2, q_3) we can write

$$ds^2 = h_1^2(dq_1)^2 + h_2^2(dq_2)^2 + h_3^3(dq_3)^2\,, \quad \text{where} \quad h_i = \left\| \frac{\partial \boldsymbol{x}}{\partial q_i} \right\|\,. \tag{3.2}$$

In letting $(q_1, q_2, q_3) = (r, \theta, z)$ and performing these simple calculations, we get the new infinitesimal line element on the cone

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2\,,$$

however using the parameterisation of z in (3.1), we have that $dz = \cot\alpha\, dr$ and so substituting this in, and using the fact that $1 + \cot^2\alpha = \csc^2\alpha$ we have the correct form of our infinitesimal line element,

$$ds^2 = \csc^2(\alpha) dr^2 + r^2 d\theta^2\,. \tag{3.3}$$

Part b)

We can now use the parametrisation $r = r(\theta)$ for $\theta \in [\theta_0, \theta_1]$. To find the geodesics on a cone, we wish to minimise the arc length functional on the curve $r = r(\theta)$, namely

$$S[r(\theta), r'(\theta)] = \int ds = \int \sqrt{\csc^2(\alpha) dr^2 + r^2 d\theta^2} = \int_{\theta_0}^{\theta_1} \underbrace{\sqrt{\csc^2(\alpha) r'(\theta)^2 + r^2}}_{f(r(\theta), r'(\theta))}\, d\theta\,. \tag{3.4}$$

Part c)

Since $f(r(\theta), r'(\theta))$ does not *explicitly* depend on θ, i.e. θ is an ignorable coordinate, we can use the Beltrami equation,

$$f - r'\frac{\partial f}{\partial r'} = C\,, \tag{3.5}$$

where C is a constant, to simplify our calculations. We can then calculate

$$\frac{\partial f}{\partial r'} = \frac{\partial}{\partial r'}(\csc^2(\alpha)r'(\theta)^2 + r^2)\left(\frac{1}{2\sqrt{\csc^2(\alpha)r'(\theta)^2 + r^2}}\right) = \frac{\csc^2(\alpha)r'(\theta)}{\sqrt{\csc^2(\alpha)r'(\theta)^2 + r^2}}\,,$$

and so plugging this into (3.5), we see that

$$\sqrt{\csc^2(\alpha)r'(\theta)^2 + r^2} - \frac{\csc^2(\alpha)r'(\theta)^2}{\sqrt{\csc^2(\alpha)r'(\theta)^2 + r^2}} = \frac{r^2}{\sqrt{\csc^2(\alpha)r'(\theta)^2 + r^2}} = C\,. \quad (3.6)$$

Rearranging for $r'(\theta)$ we arrive at the differential equation governing the form of geodesics on the cone,

$$\frac{dr}{d\theta} = r'(\theta) = \frac{r}{\csc\alpha}\sqrt{\left(\frac{r}{C}\right)^2 - 1}\,. \quad (3.7)$$

Part d)

In order to now solve this differential equation, we first integrate and then use the constraint that $r(\theta)$ must pass through $(r_0, \pm\Delta\theta)$ in order to solve for the constants. By integrating both sides of (3.7) we get

$$\int \sin(\alpha)d\theta = \int \frac{1}{r\sqrt{\left(\frac{r}{C}\right)^2 - 1}}dr\,,$$

and by using the identity provided to us in the question, we see that

$$\sin(\alpha)\theta + A = \operatorname{arcsec}(r/C)\,,$$

for arbitrary constants A and C, which leads us to the general form of a geodesic on a cone (assuming principal values - more on this later):

$$\boxed{r(\theta) = C\sec(\sin(\alpha)\theta + A)}\,. \quad (3.8)$$

Plugging in our constraints we get the simultaneous equations

$$r_0 = C\sec(A + \sin(\alpha)\Delta\theta)\,, \quad \text{and} \quad r_0 = C\sec(A - \sin(\alpha)\Delta\theta)\,. \quad (3.9)$$

Letting these equal one another, with appropriate rearrangement and use of trigonometric identities to simplify $\cos(w - z) = \cos(w + z)$, this yields

$$\sin(A)\sin(\sin(\alpha)\Delta\theta) = 0\,, \quad (3.10)$$

and since this must be true for all values of $\Delta\theta$, and *assuming that* $\alpha \neq 0$ (which is violated in part f), this gives us that $A = n\pi$ for $n \in \mathbb{Z}$. We can then solve for C in (3.9):

$$C = r_0\cos(n\pi + \sin(\alpha)\Delta\theta) = (-1)^n r_0\cos(\sin(\alpha)\Delta\theta)\,, \quad (3.11)$$

and so substituting these into (3.8), we calculate

$$\begin{aligned} r(\theta) &= (-1)^n r_0 \cos(\sin(\alpha)\Delta\theta)\sec(\sin(\alpha)\theta + n\pi) \\ &= (-1)^n r_0 \cos(\sin(\alpha)\Delta\theta)(-1)^n \sec(\sin(\alpha)\theta)\,. \end{aligned}$$

Thus, we arrive at the curve of the geodesic connecting the two points $(r_0, \pm\Delta\theta)$,

$$\boxed{r(\theta) = (r_0 \cos(\sin(\alpha)\Delta\theta))\sec(\sin(\alpha)\theta)}\,. \tag{3.12}$$

As a brief example, consider setting $\alpha = \frac{\pi}{6}$, $\Delta\theta = \frac{\pi}{2}$, and $r_0 = \sqrt{2}$ which would lead to $r(\theta) = \sec(\theta/2)$. We could then use a 3D parametric plot on

$$(x(t), y(t), z(t)) = \sec(t/2)\left(\cos(t), \sin(t), \sqrt{3}\right) \quad \text{for } t \in [-\pi/2, \pi/2]$$

to get an idea of what we have calculated.

Part e)

Throughout the previous question when manipulating trigonemetric functions, in particular when using their inverses to arrive at (3.8), we assumed the use of principal values of angles. Clearly, though, we could have found more general solutions throughout. Similarly to the cylindrical example in the notes, this leads us to believe that the geodesics are *not unique*.

Viewed in a different light, it can be shown that a cone is isometric to the plane - in fact, even a child can see this by cutting out a circular sector from paper and wrapping it around to form a cone. Due to this isometry, we can consider multiple copies of the cone on the plane, corresponding to multiple circular sectors pasted together at their radial edges. With a little bit of work, one could determine some specifics around "how many" copies are possible, i.e. how many distinct sectors could fit in the one circle before the circle is complete and the same point on the xy-plane is repeated. We leave this as an exercise for the reader.

In conclusion, these geodesics are *not unique*, however, there is only a finite amount of them (unlike the scenario for the cylinder) since we can only "roll the cone on to the plane" a finite number of times before repetition of coordinates.

Part f)

We see that for our particular solution in (3.12) we have

$$\lim_{\alpha\to 0} r(\theta) = r_0 \qquad \text{and} \qquad \lim_{\alpha\to\pi/2} r(\theta) = r_0\cos(\Delta\theta)\sec(\theta)\,. \tag{3.13}$$

The first limit makes sense as the curve is simply a constant between two points of the same radius - indeed, the cone at $\alpha = 0$ is just the z-axis, so any two points of the same radius must be the same point. For the second limit, we see that

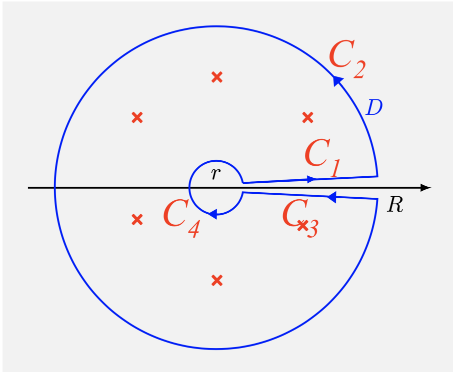
$$(x, y, z) = (r_0\cos(\Delta\theta), r_0\cos(\Delta\theta)\tan(\theta), 0)\,, \tag{3.14}$$

so the curve joining the two points is simply a straight line in the xy-plane that goes from $(x(\Delta\theta), -y(\Delta\theta)) \to (x(\Delta\theta), y(\Delta\theta))$ as θ varies from $[-\Delta\theta, \Delta\theta]$, i.e. a constant in x since the two coordinates $(r_0, \pm\Delta\theta)$ will always induce the same x-coords. We obviously expect this since the cone at $\alpha = \pi/2$ is just the xy-plane.

Q4. Improper logarithmic integral with complex analysis

Let $n \geq 2$ be an integer, $r < 1$ and $R > 1$. We want to determine the value of the integral

$$L_n = \int_0^\infty \frac{\log x}{1 + x^n} dx \,. \tag{4.1}$$



To do this, we will consider the integral of

$$J_n = \oint_D \frac{(\log z)^2}{1 + z^n} \tag{4.2}$$

along the contour $D = C_1 + C_2 + C_3 + C_4$ as given in the diagram to the right. For $z = re^{i\theta}$, we take the complex logarithm $\log z = \log r + i\theta$ to be defined on the domain $U = \{(r, \theta) : r > 0 \,, \theta \in [0, 2\pi)\}$. This means that we have a branch cut on the positive real axis.

First we will find the singularities of

$$f(z) = \frac{(\log z)^2}{1 + z^n} \,. \tag{4.3}$$

Since $\log z$ is well defined on U, we don't need to worry about any singularity contributions from the numerator, meaning the only singularities will come from finding the roots of (negative) unity in the denominator:

$$1 + z^n = 0 \,, \quad \text{so} \quad z^n = r^n e^{in\theta} = e^{(2k+1)\pi i} \,, \quad \text{so} \quad z_k = e^{\frac{(2k+1)}{n}\pi i} \tag{4.4}$$

are the simple poles of $f(z)$ for $k = 0, \ldots, (n-1)$. Notice that these occur in complex conjugate pairs, and in the case of n being odd, $z = -1$ is also included and is the only singularity that can occur on the real axis.

We can then calculate the residues of these simple poles using the fact that since we can express $f(z) = P(z)/Q(z)$ for holomorphic $P(z) = (\log z)^2$ and the polynomial $Q(z) = 1 + z^n$, and all poles are simple, then $\text{Res}(f; z_k) = P(z_k)/Q'(z_k)$. Hence,

$$\text{Res}(f, z_k) = \frac{P(z_k)}{Q'(z_k)}, = \frac{(\log e^{\frac{(2k+1)}{n}\pi i})^2}{n(e^{\frac{(2k+1)}{n}\pi i})^{n-1}} = \frac{\pi^2 (2k+1)^2}{n^3} e^{\frac{(2k+1)}{n}\pi i} \,. \tag{4.5}$$

We can then calculate the sum of these residues. Note that by hand one would use standard formulas for the likes of $\sum k^2 r^k$ to do this, but the step of evaluating these (i.e. from line 2 to 3 in the following calculation) was assisted by WolframAlpha for ease. All other steps merely involve factorising factors of $e^{\pi i/n}$ and noting standard definitions of trigonemtric functions from complex exponentials.

Enough talk - we have:

$$
\begin{aligned}
\sum_{k=0}^{n-1} \mathrm{Res}(f, z_k) &= \frac{\pi^2}{n^3} \sum_{k=0}^{n-1} (2k+1)^2 e^{\frac{(2k+1)}{n}\pi i} \\
&= \frac{\pi^2}{n^3} \left[4\sum_{k=0}^{n-1} k^2 e^{\frac{(2k+1)}{n}\pi i} + 4\sum_{k=0}^{n-1} k e^{\frac{(2k+1)}{n}\pi i} + \sum_{k=0}^{n-1} e^{\frac{(2k+1)}{n}\pi i} \right] \\
&= \frac{4\pi^2}{n^3} \left[\frac{n e^{\pi i/n} \left((n-2)e^{2\pi i/n} - n \right)}{\left(e^{2\pi i/n} - 1\right)^2} + \frac{n e^{\pi i/n}}{\left(e^{2\pi i/n} - 1\right)} + 0 \right] \\
&= \frac{4\pi^2}{n^3} \left(\frac{n e^{\pi i/n}}{e^{2\pi i/n} - 1} \right) \left(\frac{(n-2)e^{2\pi i/n} - n}{e^{2\pi i/n} - 1} + 1 \right) \\
&= \frac{4\pi^2}{n^2} \frac{\csc(\pi/n)}{2i} \left(\frac{(n-2)e^{\pi i/n} - n e^{-\pi i/n} + e^{\pi i/n} - e^{-\pi i/n}}{e^{\pi i/n} - e^{-\pi i/n}} \right) \\
&= \frac{2\pi^2}{n^2} \csc(\pi/n)(-i) \left(\frac{n\left(e^{\pi i/n} - e^{-\pi i/n}\right)}{e^{\pi i/n} - e^{-\pi i/n}} - \frac{e^{\pi i/n} + e^{-\pi i/n}}{e^{\pi i/n} - e^{-\pi i/n}} \right) \\
&= \frac{2\pi^2}{n^2} \csc(\pi/n) \left[\cot(\pi/n) - ni \right] . \qquad (4.6)
\end{aligned}
$$

Thus, by the residue theorem,

$$
\oint_D f(z)dz = 2\pi i \sum_{k=0}^{n-1} \mathrm{Res}(f, z_k) = \frac{4\pi^3 i}{n^2} \csc(\pi/n) \left[\cot(\pi/n) - ni \right] . \qquad (4.7)
$$

It remains to analyse the individual contributions C_j.

C_1 and C_3

Let C_1 be parametrised as $z = x$ for $r \leq x \leq R$, and C_3 as $z = xe^{2\pi i}$ for $R \leq x \leq r$, i.e. the reverse direction and on the other side of the branch cut. Then,

$$
\begin{aligned}
\left(\int_{C_1} + \int_{C_3} \right) f(z)dz &= \int_r^R \frac{(\log x)^2}{1+x^n} dx + \int_R^r \frac{(\log x + 2\pi i)^2}{1 + (xe^{2\pi i})^n} dx \\
&= \int_r^R \frac{(\log x)^2}{1+x^n} dx - \int_r^R \frac{(\log x)^2}{1+x^n} dx - \int_r^R \frac{4\pi i \log x}{1+x^n} dx + \int_r^R \frac{4\pi^2}{1+x^n} dx ,
\end{aligned}
$$

and then taking limits as $r \to 0$ and $R \to \infty$, and using the identity provided in the question, we arrive at

$$
\left(\int_{C_1} + \int_{C_3} \right) f(z)dz = -4\pi i L_n + \frac{4\pi^3}{n} \csc\left(\frac{\pi}{n}\right) . \qquad (4.8)
$$

C_2 (outer radius)

We aim to show that this contribution goes to 0 as $R \to \infty$. Let C_2 be parametrised as $z = Re^{i\theta}$, $dz = Rie^{i\theta} d\theta$ for $0 \leq \theta < 2\pi$. Then

$$
\int_{C_2} f(z)dz = \int_0^{2\pi} \frac{(\log R)^2 + 2i\theta \log R + (i\theta)^2}{1 + R^n e^{in\theta}} Rie^{i\theta} d\theta . \qquad (4.9)
$$

We can then bound the integrand:

$$
\begin{aligned}
|f(z)| &= \left| \frac{R(\log R)^2 + 2i\theta R \log R - R\theta^2}{1 + R^n e^{in\theta}} ie^{i\theta} \right| \\
&= \left| \frac{R(\log R)^2 + 2i\theta R \log R - R\theta^2}{1 + R^n e^{in\theta}} \right| \\
&\leq \left| \frac{R(\log R)^2 + 2i\theta R \log R}{1 + R^n e^{in\theta}} \right| && \text{since } R, \theta^2 > 0, \\
&\leq \left| \frac{R(\log R)^2}{1 + R^n e^{in\theta}} \right| + \left| \frac{2\theta R \log R}{1 + R^n e^{in\theta}} \right| && \text{by the triangle inequality,} \\
&\leq \left| \frac{R(\log R)^2}{R^n - 1} \right| + 4\pi \left| \frac{R \log R}{R^n - 1} \right| && \text{by the reverse triangle inequality.}
\end{aligned} \tag{4.10}
$$

Then, by an easy application of L'Hopital's rule (and noting that $n \geq 2$) that we omit for brevity, we can show that both of these terms $\to 0$ as $R \to \infty$. Thus by the ML-bound we can conclude that

$$
\lim_{R \to \infty} \left| \int_{C_2} f(z) dz \right| \leq \lim_{R \to \infty} 2\pi \, |f(z)| = 0 \tag{4.11}
$$

as desired.

<u>**C_4 (inner radius)**</u>

Again, we will show this contribution goes to 0 in the limit. Let C_4 be parametrised as $z = re^{i\theta}$ for $2\pi < \theta \leq 0$. Then $\int_{C_4} f(z)dz$ expands in exactly the same way as in (4.9) where the terminals are merely reversed and we replace R with r. Then we can arrive at an identical bound for $|f(z)|$ along this arc as in (4.10). Then using the standard limit

$$
\lim_{r \to 0} r \log r = 0 \,, \tag{4.12}
$$

we can use L'Hopitals rule on the first term again and (4.12) on the second term, we see that once again we have

$$
\lim_{r \to \infty} \left| \int_{C_4} f(z) dz \right| \leq \lim_{r \to \infty} 2\pi \, |f(z)| = 0 \,. \tag{4.13}
$$

<u>**Conclusion**</u>

Putting all of this together, in particular combining the residues from (4.7) and the contributions from (4.8), we have

$$
\oint_D f(z) dz = -4\pi i L_n + \frac{4\pi^3}{n} \csc\left(\frac{\pi}{n}\right) = \frac{4\pi^3 i}{n^2} \csc(\pi/n) \left[\cot(\pi/n) - ni\right] \,, \tag{4.14}
$$

and so after dividing by $-4\pi i$ and taking the real parts of both sides, we finally arrive at the glorious solution,

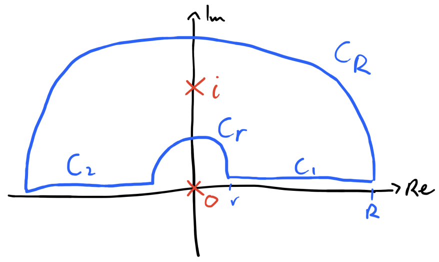
$$
\boxed{L_n = -\frac{\pi^2}{n^2} \csc(\pi/n) \cot(\pi/n)} \,. \tag{4.15}
$$

We note that next time it would be much easier to take a contour that only contained one simple pole - that damn residue calculation was a pain. □

Q5. A trigonemetric improper integral

We want to compute

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)} dx \,, \qquad (5.1)$$



and will do so by evaluating

$$\oint_C \frac{e^{iz}}{z(1+z^2)} dz \qquad (5.2)$$

around the contour to the right. (Apologies for the diagram looking like it was drawn in crayon). With suitable parametrisations, we have

$$\oint_C \frac{e^{iz}}{z(1+z^2)} dz = \left(\int_r^R + \int_{-R}^{-r} \right) \frac{e^{ix}}{x(1+x^2)} dx + \int_{C_R} \frac{e^{iz}}{z(1+z^2)} dz + \int_{C_r} \frac{e^{iz}}{z(1+z^2)} dz \,, \qquad (5.3)$$

where we note that for the first two contributions we have, as $R \to \infty$ and $r \to 0$,

$$\text{P.V. Im} \left(\int_r^R + \int_{-R}^{-r} \right) \frac{e^{ix}}{x(1+x^2)} dx = I \,. \qquad (5.4)$$

For the contribution C_R, consider $g(z) = \frac{1}{z(1+z^2)}$. On the arc $C_R = \{z : |z| = R\}$, we have by the reverse triangle inequality for $R > 1$

$$|g(z)| \leq \frac{1}{R(R^2-1)} = M_R \,, \qquad (5.5)$$

which means that $g(z)$ tends uniformly to 0 on the arc C_R as $R \to \infty$ since M_R does not depend on θ, causing the uniformity. Hence, by Jordan's Lemma for the 1st and 2nd quadrants, we have

$$\lim_{R\to\infty} \int_{C_R} e^{iz} g(z) dz = 0 \,, \qquad (5.6)$$

so the C_R contribution in (5.3) goes to 0 in the limit.

Next consider the C_r contribution, the indented contour around $z = 0$. By the "Limiting Contours IV" theorem in lectures, since $z = 0$ is a simple pole and the circular contour is centred around $z = 0$ by an arc of angle $\alpha = \pi$, we have

$$\lim_{r\to 0} \int_{C_r} \frac{e^{iz}}{z(1+z^2)} dz = -i\pi \text{Res}(f; 0) = -i\pi \lim_{z\to 0} \frac{ze^{iz}}{z(1+z^2)} = -i\pi \,, \qquad (5.7)$$

where the negative arises due to the clockwise direction.

Finally we calculate the residue at the simple pole $z = i$ inside the contour,

$$\text{Res}(f; i) = \lim_{z\to i} \frac{(z-i)e^{iz}}{z(1+z^2)} = -\frac{e^{-1}}{2} \,. \qquad (5.8)$$

Putting all of this together in taking $r \to 0$ and $R \to \infty$, we have

$$\oint_C \frac{e^{iz}}{z(1+z^2)} dz = \left(\int_0^{\infty} + \int_{-\infty}^{0} \right) \frac{e^{ix}}{x(1+x^2)} dx + i\pi = -\pi i e^{-1}\,, \tag{5.9}$$

and so taking imaginary parts and principal values we arrive at

$$\boxed{\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)} dx = (1 - e^{-1})\pi}\,. \tag{5.10}$$